17/10/25

TM253637S

Reg. No :.....

Name :.....

MASTER'S DEGREE (C.S.S) EXAMINATION, NOVEMBER 2025

2024 ADMISSIONS REGULAR

SEMESTER III - CORE COURSE

MT3C12TM20 - Functional Analysis

Time : 3 Hours

Maximum Weight : 30

Part A

I. Answer any Eight questions. Each question carries 1 weight (8x1=8)

1. Prove that every convergent sequence in a normed space is bounded.
2. If two norms $\|.\|$ *and* $\|.\|_0$ on a vector space are equivalent, show that $\|x_n - x\| \to 0 \Rightarrow \|x_n - x\|_0 \to 0$ and viceversa.
3. If T: X $\to$Y is continuous and M$\subset$ X is compact. Prove that T(M) is compact.
4. Let f be a linear functional on n- dimensional vector space X. Find the dimension of N(f).
5. In a real inner product space X, prove that $< x, y > = \frac{1}{2}(\|x+y\|^2 + \|x-y\|^2)$.
6. Define orthogonal complement of a Hilbert space H. Prove that $Y^{\perp}$ is a closed subspace of H.
7. Let X be an inner product space and M$\subset$X. Prove that M is total in X implies $M^{\perp}$ = {0}. Is the converse true? Justify.
8. Show that if T is a bounded self adjoint linear operator on a Hilbert space H so is T^n, where n is a positive integer.
9. If p_1 and p_2 are sulinear functionals on vector space X and c_1 and c_2 are positive constants. Show that p = c_1 p_1+ c_2 p_2 is sublinear.
10. Prove that

    (a) $(S+T)^x = S^x + T^x$ (b)) $(\alpha T)^x = \alpha T^x$, $\forall$ S,T $\in$ B(X,Y)

Part B

II. Answer any Six questions. Each question carries 2 weight (6x2=12)

11. Prove that any norm is equivalent to any other norm on a finite dimensional vector space.
12. Define a bounded linear operator. Also define ||T|| and prove that ||T|| satisfies all the conditions of norm.
13. Give an example of a normed space which is not an inner product space.
14. a) If in an inner product space $x_n \to x$ and $y_n \to y$, prove that $<x_n, y_n> \to <x, y>$ b) Let T: X$\to$ X be a bounded linear operator in a complex inner product space X if <Tx, x> = 0 $\forall$ x ε X , show that T = 0.
15. Let X be an inner product space and M be a complete subspace of X. Prove that for a fixed x in X there exist y in M such that $z = (x-y) \perp M$
16. Let X and Y be inner product spaces and Q: X $\to$ Y a bounded linear operator. Then prove that

    (a) Q = 0 if and only if < Qx, y> = 0 for all x ε X and y ε Y.

    (b) Q: X $\to$ X, where X is complex and < Qx, x> = 0 for all x ε X then Q = 0.

17. Let X be a normed space and x_0 be any non zero element of X. Prove that there exists a bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = \frac{1}{\|x_0\|}$ *and* $\tilde{f}(x_0) = 1$.

18. If a normed space X is reflexive then prove that X' is also reflexive.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. Prove that every finite dimensional subspace of a normed space is complete.

20. a) Define dual space of a normed space

    b) Prove that dual space of l^1 is l^∞ .

21. a) Explain Gram-Schmidt process of orthonormalisation. b) Orthonormalise 1st three terms of the sequence$\{x_0,x_1,x_2....\}$ where $x_t(j) = t^j$ on the interval [-1, 1] with respect to the inner product given by

$$< x,y > = \int_{-1}^{1} x(t)y(t)dt.$$

22. State and prove Hahn Banach theorem for complex vector spaces.