TM253287B

Reg. No : 31/10/25

Name :.....

MASTER'S DEGREE (C.S.S) EXAMINATION, OCTOBER 2025

2024 ADMISSIONS REGULAR

SEMESTER III - CORE COURSE

MT3C14TM20 - Differential Geometry

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Consider the level set $f^{-1}(-1), f^{-1}(0)$ and $f^{-1}(1)$ for $f(x_1, x_2, x_3, \dots, x_{n+1}) = {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 - {x_{n+1}}^2$, n = 1, 2. Find which points 'p' of these level sets fail to have tangent spaces equal to $(\nabla f(p))^{\perp}$

2. Sketch the level set for n = 0,1,2 for the function $f(x_1, x_2, \dots, x_{n+1}) = x_{n+1}, c = -1,0,1,2.$

3. Let $\bar{X}$ and $\bar{Y}$ be smooth vector fields tangent to S along $\propto$, $\propto: I \to S$ and f be a smooth function along $\propto$, then prove that $(\bar{X} + \bar{Y})' = \bar{X}' + \bar{Y}'$

4. Let $\bar{X}$ and $\bar{Y}$ be smooth vector fields tangent to S along $\propto$, $\propto: I \to S$ and f be a smooth function along $\propto$, then prove that $(\bar{X}.\bar{Y})' = \bar{X}.\bar{Y}' + \bar{Y}.\bar{X}'$

5. Let $\bar{X}$ and $\bar{Y}$ be smooth tangent vector fields on S and $f: S \to R$ is a smooth function on S then prove that $D_{\bar{v}}(f\bar{X}) = \nabla_{\bar{v}} f.\bar{X} + f(p).D_{\bar{v}}\bar{X}$

6. Define Directiona derivative and Co-variant derivative of $\bar{X}$ with respect to $\bar{v}$

7. Let $\bar{X}$ and $\bar{Y}$ be smooth vector fields on U (or on S) then prove that $\nabla_{\bar{v}}(\bar{X} + \bar{Y}) = \nabla_{\bar{v}}\bar{X} + \nabla_{\bar{v}}\bar{Y}$

8. (i) Define principle curvature of S at p. (ii) Define principle curvature directions of S at p.

9. Find the normal curvature κ(v) for each tangent direction v, the principal curvatures and principal curvature directions, and the Gauss- Kronecker and mean curvatures at the given point p of the given n-surface $f(x_1, x_2, \dots, x_{n+1}) = c$ oriented by $\frac{f}{\|\nabla f\|}$ where ${x_1}^2 + \left(\sqrt{{x_2}^2 + {x_3}^2} - 2\right)^2 = 1, p = (0,3,0).$

10. Define

    i. The image of $d\Psi_p$

    ii. The tangent space of Ψ at p.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Let $a, b, c \in R$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = {x_1}^2 + {x_2}^2$ on the ellipse $a{x_1}^2 + 2bx_1x_2 + c{x_2}^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ where λ_1 and λ_2 are the Eigen values of the matrix

12. Define (i) Tangent vector field on S. (ii) Normal vector field on S

13. Give an example of a parameterized curve having constant speed but it is not a geodesic.

14. State and prove any 3 properties of Levi - Civita Parallel.

15. Let $g: I \to R$ be a smooth function and let C denote the graph of g. Show that the curvature of C at the point (t, g(t)) is $g''(t) \Big/ (1 + g'(t)^2)^{3/2}$, for an appropriate choice of orientation.

16. Prove that for each 1-form w on U open in $\mathbb{R}^{n+1}$ there existe unique functions $f_i: U \to \mathbb{R}$. i=1,2,... ,n+1 . such that $w = \sum_{i=1}^{n+1} f_i dx_i$ .Also prove that w is smooth if and only if each f_i is smooth.

17. Show that the mean curvature at a point p of an oriented n-surface S can be computed from the values of normal curvature on any orthonormal basis $\{v_1, v_2, \dots, v_n\}$ for ζp by the formula $H(p) = \frac{1}{n}\sum_{i=1}^{n} k(v_i)$.

18. Prove that on each component oriented n-surface S in R^{n+1} there exists a point p such that the second fundamental form at p is definite.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. State and prove the existence and uniqueness for maximal integral curve on a smooth vector field $\overline{\overline{X}}$.

20. Prove that, Let S be a compact connected oriented n-surface in R^{n+1} exhibited as a level set $f^{-1}(c)$ of a smooth function $f: R^{n+1} \to R$ with $\nabla f(p) \neq 0 \forall\, p \in S$, then gauss map maps S onto the unit sphere S^n.

21. Compute the Weingarten map for

    i. The Hyper plane, $a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} = b, (a_1, a_2, \dots, a_{n+1}) \neq (0,,0,0, \dots 0).$

    ii. The circular Cylinder, ${x_2}^2 + {x_3}^2 = a^2, in\ R^3, (a \neq 0).$

22.

    i. Prove that Let S be a compact connected oriented n-surface in R^{n+1}, then the Gauss-Kronecker curvature K(p) of S at p is non-zero for all $p \in S$ if and only if the second fundamental form ζp of S at p is definite for all $p \in S$.

    ii. Prove that on each component oriented n-surface S in R^{n+1} there exists a point p such that the second fundamental form at p is definite.