TM253644G

Reg. No :.....

Name :.....

MASTER'S DEGREE (C.S.S) EXAMINATION, NOVEMBER 2025

2024 ADMISSIONS REGULAR

SEMESTER III - CORE COURSE

MT3C11TM20 - Advanced Abstract Algebra

Time : 3 Hours

Maximum Weight : 30

Part A

I. Answer any Eight questions. Each question carries 1 weight (8x1=8)

1. Define (i) Finite Extension (ii) Algebraic Extension
2. Find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.
3. Define UFD and PID.
4. Let R be a commutative ring and let $N_1 \subseteq N_2 \subseteq \ldots$ be an ascending chain of ideals N_i in R. Prove that $N = \bigcup N_i$ is an ideal of R.
5. Let $\{\sigma_i : i \in I\}$ be the collection of automorphisms of a field E then prove that the set $E_{\{\sigma_i\}}$ of all $a \in E$ left fixed by every $\sigma_i, i \in I$ forms a subfield of E.
6. i. Define splitting field over F.

   ii. Identify the splitting field of $\{x^2 - 2, x^2 - 3\}$ and $\{x^4 - 5x^2 + 6\}$.
7. What is the order of $G(Q(\sqrt[3]{2}, i\sqrt{3})/Q(\sqrt[3]{2}))$.
8. When an irreducible polynomial is separable over a field F? Give an example.
9. Define separable extension of F. Give example.
10. Define finite normal extension of F.

Part B

. Answer any Six questions. Each question carries 2 weight (6x2=12)

1. Prove that the multiplicative group $<F^*, .>$ of non-zero elements of a finite field F is cyclic.
2. Find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.
3. If D is a PID, prove that every element that is neither 0 nor a unit in D is a product of irreducibles.
4. Define multiplicative norm on D. If D is an integral domain with a multiplicative norm N, prove that N(1) = 1 and | N(u) | = 1 for every unit u in D. Furthermore if for every α such that | N(α) | = 1 is a unit in D, prove that an element π in D, with | N(π) | = p for a prime p in $\mathbb{Z}$, is an irreducible of D.
5. Prove that the number of field extensions is completely determined by the two fields E and F.
6. Prove that a field E, where $F \leq E \leq \bar{F}$, is a splitting field over F if and only if every automorphism of $\bar{F}$ leaving F fixed maps E onto itself and thus induces an automorphism of E leaving F fixed.
7. Prove that if E is a finite extension of F, then { E:F } divides [E:F].
8. a) Let K be a finite extension of degree n of a finite field f of p^r elements. Prove that G(K/F) is cyclic of order n and is generated by σ_α, where $\alpha \in K$, $\sigma_\alpha(\alpha) = \alpha^{p^r}$

$(\bar{-}, +)$.

Part C

Answer any Two questions. Each question carries 5 weight **(2x5=10)**

i. Let E be a simple extension $F(\alpha)$ of a field F, and let α be algebraic over F. Let the degree of $irr(\alpha, F)$ be $n \geq 1$. Then prove that every element β of $E = F(\alpha)$ can be uniquely expressed in the form $\beta = b_0 + b_1 \alpha + b_2 \alpha^2 + b_3 \alpha^3 + \cdots + b_{n-1} \alpha^{n-1}$,where the b_i are in F.

ii. Construct a field of 9 elements with $p(x) = x^2 - x - 2$. Also identify the multiplicative inverse of each elements.

a) Prove that if D is a UFD; prove that D[x] is a UFD.

b) If F is a field and $x_1, x_2 \ldots\ldots x_n$ are indeterminates, prove that $F[x_1, x_2 \ldots\ldots x_n]$ is a UFD.

Let $\bar{F}$ and $\bar{F}'$ be two algebraic closures of F the prove that $\bar{F}$ is isomorphic to $\bar{F}'$ under an isomorphism leaving each element of F fixed.

a) Define perfect field. Prove that every field of characteristic zero is perfect. b) Prove that every finite field is perfect.