Project Report

on

A STUDY ON RHOTRIX THEORY

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CERTIFICATE

This is to certify that the dissertation entitled, **A STUDY ON RHOTRIX THEORY** is a bonafide record of the work done by **CHRIS CATHERIN CYRIAC** under my guidance as partial fulfillment of the award of the degree of **Bachelor of Science in Mathematics** at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Dr. Elizabeth Reshma M T , Assistant Professor , Department of Mathematics , St. Teresa's College (Autonomous) , Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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Chapter 1

INTRODUCTION TO RHOTRIX THEORY

1.1 HISTORY OF RHOTRIX THEORY

The theory of rhotrix is a relatively new area of mathematical discipline dealing with algebra and analysis of arrays of numbers in mathematical rhomboid form. The theory began from the work of (Ajibade, 2003), when he initiated the concept, algebra and analysis of rhotrices as an extension of ideas on matrix-tersions and matrix-noitrets proposed by (Atanassov and Shannon, 1998). Ajibade gave the initial definition of a rhotrix of size 3 as a mathematical array that is in some way, between two-dimensional vectors and 2×2 dimensional matrices. Since the introduction of the theory in 2003, many authors have shown interest in the usage of the rhotrix set, as an underlying set, for construction of algebraic structures. [7]

Following Ajibade's work, (Sani, 2004) proposed an alternative method for multiplication of rhotrices of size three based on their rows and columns as comparable to matrix multiplication, which was considered to be an attempt to answer the question of "whether a transformation can be made to convert a matrix into a rhotrix and vice versa" posed in the concluding section of the initial article on rhotrix. This method of multiplication is now referred to as "row-column based method for rhotrix multiplication". Unlike Ajibade's method of multiplication that is both commutative and associative, Sani's method of rhotrix multiplication is non-commutative but associative. [15]

1.2 CONCEPT OF RHOTRIX IN MATHEMATICAL ENRICHMENT

A rhotrix is a rhomboid array of numbers where c = h(R) is called the heart of any rhotrix R and R is the set of real numbers. R is the set of all 3 dimensional rhotrices.

Heart of a Rhotrix : The element at perpendicular intersection of the two diagonals of any rhotrix is known as the heart of the rhotrix.

Dimension : A rhotrix is always of odd dimension.

Cardinality : A rhotrix of dimension n has $(n^2+1)/2$ entries.

Coupled Matrix : The transpose of a matrix is an operator which flips a matrix over its diagonal, that is equivalent to rotating columns of the matrix by 90° in anticlockwise direction. The coupled matrices of Rhotrix can be obtained by rotating its columns by 45 degrees in anticlockwise direction instead of 90 degrees.

Let's consider a rhotrix of order 5.

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle \qquad R_5^{\frac{T}{2}} = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{12} & \\ a_{21} & & a_{22} & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix}$$

Here T/2 indicates half rotation in comparison to transpose, but direction is the same (anti-clock wise). We observe two coupled matrices– higher order matrix A known as major matrix and lower order matrix C is known as minor matrix.

$$\boldsymbol{A}_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \boldsymbol{C}_{2\times2} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

In general, for a Rhotrix of order n (R{n}) one can have

$R_n^{T/2} = \langle a_{ij}\ ,\ c_{lk} \rangle^{T/2} = [\boldsymbol{A}_{ij}\ \ \boldsymbol{C}_{lk}] = [\boldsymbol{A}\ \boldsymbol{C}]$

(say),where i,j = 1,2 ,3 ...,t ; l,k = 1,2,3..t. So, A and C are coupled square matrices of order t and t-1,where t=(n+1)/2.[16]

Major And Minor Entries :

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an n-dimensional rhotrix. Then, (a_{ij}) is the (i,j) entries and are called the major entries of R_n and c_{kl} is the (k,l)-entries and are called the minor entries of R_n.

Major And Minor Matrices :

A rhotrix $R_n = \langle a_{ij}, c_{kl} \rangle$ of n-dimension is a coupled of two matrices (a_{ij}) and (c_{kl}) consisting of its major and minor matrices respectively. Therefore, (a_{ij}) and (c_{kl}) are the major and minor matrices of R_n.

Major And Minor Rows :

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an n-dimensional rhotrix. Then, rows and columns of (c_{ij}) and c_{kl}) will be called the major (minor) rows and columns of R_n respectively. [4]

Transpose of a Rhotrix : The transpose of a Rhotrix can be obtained by exchanging rows with corresponding columns of a Rhotrix. Let, R be a rhotrix of order 3.

$$R_3 = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \text{ Then, } \quad R^T = \left\langle \begin{array}{ccc} & a & \\ d & h(R) & b \\ & e & \end{array} \right\rangle$$

The transpose of two Rhotrices R and Q of same order hold the property $(RQ)^T = Q^T * R^T$.

Example:

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 4 & 3 & 6 & \\ 7 & 7 & 5 & 0 & 3 \\ & 1 & 2 & 3 & \\ & & 9 & & \end{array} \right\rangle \text{ Then its transpose is, } \quad (R)^T = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 6 & 3 & 4 & \\ 3 & 0 & 5 & 7 & 7 \\ & 3 & 2 & 1 & \\ & & 9 & & \end{array} \right\rangle$$

Determinant of a Rhotrix : Let $R_3 = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ with the coupled matrices of R_3 as $\boldsymbol{A} = \begin{bmatrix} a & d \\ b & e \end{bmatrix}$

Then the determinant of $|R_3| = |A|\ |C| h(R)(ae-bd)$. Thus, the product of determinants of the coupled matrix is the determinant of the rhotrix itself.[16]

Index Based Representation of an Arbitrary Rhotrix : Unlike in an arbitrary matrix where the indices of an entry can uniquely be identified by the rows and columns of the matrix, there are various ways of representing the entries of an arbitrary rhotrix. Usually, an author would employ a particular method to suit the usage of the object.

One way of representing an arbitrary rhotrix is by the use of a single index for each entry as in the following:

$$\left\langle \begin{array}{ccccc} & & a_1 & & \\ & a_6 & a_4 & a_2 & \\ a_{11} & a_9 & a_7 & a_5 & a_3 \\ & a_{12} & a_{10} & a_8 & \\ & & a_{13} & & \end{array} \right\rangle.$$

This method is called the single-index method of arbitrary rhotrix representation. The indices can also be allowed to run horizontally from left to right.

Another way is to use two indices, the first indicating the row in which the entry lies, and the second indicating the column in which the entry lies as in the following rhotrix. See for the definitions of a row and a column of a rhotrix.

$$\left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{31} & a_{22} & a_{13} & \\ a_{51} & a_{42} & a_{33} & a_{24} & a_{15} \\ & a_{53} & a_{44} & a_{35} & \\ & & a_{55} & & \end{array} \right\rangle.$$

This method is called row-column method of representing the entries of an arbitrary rhotrix since each entry can be identified by its row and column. In this method the location (or position) of a row in a rhotrix is identified by the first index in its entries while that of a column is identified by the second index in its entries. Thus, the first row has 1 as the first index in its entries while the third column has 3 as the second index in its entries and so on. It is important not to confuse this with row-column multiplication of rhotrices.

A third method also uses two indices for each entry, where the first index indicates the row in which the entry lies. However, the second index does not indicate the column in which the entry lies in the rhotrix. This type of rhotrix representation can be seen in the following rhotrix:

$$\left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{31} & a_{21} & a_{12} & \\ a_{51} & a_{41} & a_{32} & a_{22} & a_{13} \\ & a_{52} & a_{42} & a_{33} & \\ & & a_{53} & & \end{array} \right\rangle.$$

It is this method of arbitrary rhotrix representation that is our focus in this paper, and is termed row-wise method of arbitrary rhotrix representation.[9]

1.3 BASIC PROPERTIES OF RHOTRICES

1.3.1 OPERATIONS OF RHOTRICES

Addition of Rhotrices : Only two rhotrices having the same dimension can be added together. The sum of the matching elements of two rhotrices is the definition of their addition. Let R_3 and Q_3 be two 3-dimensional rhotrices such that,

$$R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \text{ and } \mathbf{Q_3} = \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle$$

Then their addition is defined as

$$R_3 + Q_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle . + \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & x+u & \\ s+q & v+y & p+w \\ & t+z & \end{array} \right\rangle$$

Scalar Multiplication of Rhotrix : In scalar multiplication, the given scalar is multiplied by each entry in a rhotrix.

Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$

and α be a scalar number. Then the scalar multiplication of a rhotrix is defined as

$$\alpha(R3) = \alpha \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \alpha x & \\ \alpha q & \alpha y & \alpha p \\ & \alpha z & \end{array} \right\rangle$$

Multiplication of Rhotrix : Rhotrices can be multiplied in two different ways. Row-column multiplication and heart-oriented multiplication are these. While row-column multiplication resembles matrix multiplication, heart- oriented multiplication is connected to the heart of a rhotrix.

Heart Oriented Multiplication : As the name implies, we multiply each element of the first rhotrix by the heart of the second rhotrix, and we add what comes out to the product of the corresponding element of the second rhotrix and the heart of the first rhotrix.

Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ and $Q_3 = \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle$

Let R and Q be two non-zero rhotrices, then from above result, we have

$$R_3 o Q_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & u & \\ s & v & w \\ & t & \end{array} \right\rangle \quad \left\langle \begin{array}{ccc} & xv + uy & \\ qv + sy & yv & pv + wy \\ & zv + ty & \end{array} \right\rangle$$

Identity Rhotrix for Heart-Oriented Multiplication

The definition of the 3-dimensional identity rhotrix is $I_3 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$

Here, I_3 is derived as follows. Let $I_3 = \left\langle \begin{array}{ccc} & m & \\ n & o & d \\ & r & \end{array} \right\rangle$ be the identity rhotrix and

$R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ be a rhotrix,where $h(Rn) \neq 0$ Since $R_3 o I_3 = I_3 o R_3 = R_3$ we have

$$\left\langle \begin{array}{ccc} & m & \\ r & o & f \\ & n & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & my + xo & \\ qo + ry & oy & fy + po \\ & zo + ny & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$$

According to the definition of rhotrice equality, we have

$$my + xo = x$$
$$qo + ry = q$$
$$oy = y$$
$$fy + po = p$$
$$zo + ny = z$$

By solving the above equations we get

$$m = n = d = r = 0 \text{ and } o = 1$$

Therefore, we obtain $I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & m & \\ r & o & f \\ & n & \end{array} \right\rangle$

Rhotrix Inverse in Heart-Oriented Multiplication

Let $h(R) \neq 0$. and R be a 3-dimensional rhotrix. If a rhotrix P exists such that RoQ = PoR = I then P is referred to as R inverse. Now, we can get a rhotrix's inverse by doing the following:

Let $R = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$

be a 3 dimensional rhotrix such that $y \neq 0$

$P = \left\langle \begin{array}{ccc} & m & \\ r & o & f \\ & & \end{array} \right\rangle$ is the inverse of P such that

$$\left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & m & \\ r & o & f \\ & n & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

$$\left\langle \begin{matrix} & my+xo & \\ qo+ry & oy & fy+po \\ & zo+ny & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$$

By definition of equality of rhotrices, we get

my + xo = 0

qo + ry = 0

oy = 1

fy + po = 0

zo + ny = 0

It follows from that o =1/y, m = $-x/y^2$, qr = $-q/y^2$, s = $-p/y^2$ and t = $-z/y^2$

Therefore we have

$$R = P^{-1} = \frac{-1}{y^2} \left\langle \begin{matrix} & x & \\ q & -y & p \\ & z & \end{matrix} \right\rangle$$

A rhotrix If h(R) = 0, then R is invertible.

Proof : If R is invertible then there exists a rhotrix P such that RoP = I.

h(RoP) = h(I)

then h(R)h(P) = 1

h(P) = $\frac{1}{h(R)}$

h(R) ≠ 0.

Remark : . Heart-oriented rhotrix multiplication is a group with respect to the set of all invertible 3-dimensional rhotrices over R.

Proof: Let Q = $\left\{ R = \left\langle \begin{matrix} & x & \\ q & y & p \\ & z & \end{matrix} \right\rangle ; y \neq 0, x, q, p, z \in \mathbb{N} \right\}$

Let R = $\left\langle \begin{matrix} & x & \\ q & y & p \\ & z & \end{matrix} \right\rangle$ and S = $\left\langle \begin{matrix} & m & \\ q & n & p \\ & o & \end{matrix} \right\rangle$

be two elements in Q. Then

$$RoQ = \left\langle \begin{matrix} & xn+my & \\ qn+uy & yn & pn+vy \\ & zn+oy & \end{matrix} \right\rangle$$

It is evident that the values of y, n, and yn differ from zero. As a result, under heart-oriented multiplication, the set Q is closed. Again for any R, S, A ∈ S we have

$$A = \left\langle \begin{array}{ccc} & d & \\ e & f & g \\ & h & \end{array} \right\rangle$$

$$Ro(SoA) = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle o \left\{ \left\langle \begin{array}{ccc} & m & \\ u & n & v \\ & o & \end{array} \right\rangle o \left\langle \begin{array}{ccc} & d & \\ e & f & g \\ & h & \end{array} \right\rangle \right\}$$

$$= \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle o \left\langle \begin{array}{ccc} & mydn & \\ uy+qn & yn & vy+gn \\ & oy+zn & \end{array} \right\rangle$$

$$= (RoS)oP$$

Thus, in the set Q, the heart-oriented multiplication operation is associative. Also,

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$ is the identity of an element of S.

Additionally, the fact that each element of S is invertible suggests that the set S is a group under the multiplication with a heart orientation. [7]

Row-Column Multiplication : B. Sani discussed row-column multiplication of rhotrices, which is an alternate technique for multiplying rhotrices. By multiplying each row of the first rhotrix by each column of the second rhotrix, each element in this approach is obtained.

Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle$ and $Q_3 = \left\langle \begin{array}{ccc} & l & \\ o & m & p \\ & n & \end{array} \right\rangle$

be two rhotrices. Then the row-column multiplication of rhotrices R_3 and Q_3 is given by,

$$R_3oQ_3 = \left\langle \begin{array}{ccc} & x & \\ q & y & p \\ & z & \end{array} \right\rangle o \left\langle \begin{array}{ccc} & l & \\ o & m & t \\ & n & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & xl+po & \\ ql+zo & my & xt+pn \\ & qt+zn & \end{array} \right\rangle$$

Identity Rhotrix Under Row-Column Multiplication

Let $I_3 = \left\langle \begin{array}{ccc} & l & \\ o & m & t \\ & n & \end{array} \right\rangle$

be the identity rhotrix under multiplication defined. Then, for any rhotrix R_3 we must have

$$R_3 o I_3 = I_3 o R_3 = R_3$$

Let $R_3 = \left\langle \begin{array}{ccc} & e & \\ f & g & h \\ & i & \end{array} \right\rangle$ c ≠ 0 Then we have

$$\left\langle \begin{array}{ccc} & el+ho & \\ fl+io & gm & et+fn \\ & ft+in & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & e & \\ f & g & h \\ & i & \end{array} \right\rangle$$

According to the definition of rhotrice equality, we obtain

$$el + ho = e$$
$$fl + io = f$$
$$gm = g$$
$$et + fn = h$$
$$f t + in = i$$

It derives from above that l = o = n = 0, o = t = 0 provided g(ei − fh) ≠ 0. Hence,

$$I_3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

Inverse of a Rhotrix Under Row-Column Multiplication :

The inverse of a rhotrix Q_3 is referred to as such if $R_3 o Q_3 = Q_3 o R_3 = I_3$

Let $R_3 = \left\langle \begin{array}{ccc} & x & \\ y & z & k \\ & l & \end{array} \right\rangle$ and let $P_3 = \left\langle \begin{array}{ccc} & f & \\ g & h & i \\ & m & \end{array} \right\rangle$ be the inverse,then

$$\left\langle \begin{array}{ccc} & x & \\ y & z & k \\ & l & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h & i \\ & m & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

Therefore,

$$\left\langle \begin{array}{ccc} & xf+kg & \\ yf+lg & zh & xi+km \\ & yi+lm & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

According to the definition of rhotrice equality, we obtain

$$xf + kg = 1$$

$$yf + lg = 0$$

$$zh = 1$$

$$xi + km = 0$$

$$yi + lm = 1$$

It follows from above that $f = \frac{l}{xl-yk}$, $g = \frac{-y}{xl-yk}$, $h = \frac{1}{c}$, $i = \frac{-k}{xl-yk}$, $k = \frac{x}{xl-yk}$.

Therefore, $Q_3 = R_3^{-1} = \frac{1}{xl-yk}\left\langle \begin{array}{ccc} & l & \\ -y & \frac{xl-yk}{c} & -k \\ & x & \end{array} \right\rangle$ provided $c(xl - yk) \neq 0$.

Let t = (1+ n) / 2 for n∈N . By 'rhotrix' we understand an object that lies in some way between

n ×n dimensional matrices and (2n-1) × (2n-1) dimensional matrices. The diagonal rhotrix will be denoted by I and is given by

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

if the heart oriented multiplication is used, and

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

if the row column multiplication method is used.

Multiplication of Rhotrices under heart oriented multiplication is

- Commutative: $R_3 \text{ o } Q_3 = Q_3 \text{ o } R_3$
- Associative: $P_3 \text{ o } (R_3 \text{ o } Q_3) = (P_3 \text{ o } R_3) \text{ o } Q_3$
- Distributive with respect to addition: $P_3 \text{ o } (R_3 + Q_3) = P_3 \text{ o } R_3 + P_3 \text{ o } Q_3$ [15]

1.3.2 Rhotrix Vector Space

Let R_3 be the set of all 3-dimensional Rhotrices over real numbers.Then R_3 is a vector space with respect to the operations of addition and scalar multiplication of Rhotrices defined as above. We call it Rhotrix vector space and this vector space is spanned by

$$\left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle$$

As well as, these vectors are linearly independent, therefore the set

$$\left\{ \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle \right\}$$

forms a basis to the vector space R_3. The set Â of all rhotrices forms a vector space which is spanned by the following vectors(rhotrices):

$$\mathbf{I} = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \quad \mathbf{J} = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \quad \mathbf{K} = \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \quad \mathbf{L} = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle,$$

and $M = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle$

These vectors are linearly independent, therefore,the set = { I,J,K,M } forms a basis for Â

Any rhotrix

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

can be written as a linear combination of the rhotrices in S so that R= aJ + bK + h(R)I + dL + eM

Multiplication table for the basis is given thus:

o	I	J	K	L	M
I	I	J	K	L	M
J	J	0	0	0	0
K	K	0	0	0	0
L	L	0	0	0	0
M	M	0	0	0	0

[13]

1.3.3 Square root of a rhotrix

Let A be a rhotrix with positive heart, in this section we will devise a procedure for finding a square root of A. This procedure will only require heart oriented multiplication .

Given a real symmetric positive definite 3 × 3 matrix A, outline a direct procedure not involving the singular values or eigenvalues of A for computing a real symmetric positive definite 3 ×3 matrix B satisfying $B^2 = A$. Since we want to find the square root of a given rhotrix A, it is equivalent to finding another rhotrix B such that $A = B^2$. Now let

$$A = \left\langle \begin{matrix} & a_1 & \\ a_2 & h(A) & a_3 \\ & a_4 & \end{matrix} \right\rangle \text{ and } B = \left\langle \begin{matrix} & b_1 & \\ b_2 & h(B) & b_3 \\ & b_4 & \end{matrix} \right\rangle .$$

Therefore we want to find b_i for i =1,...,4 and h(B) such that

It follows that $h(B) = \sqrt{h(A)}$ and

$b_i = a_i/2\sqrt{h(A)}$, i =1,...,4

An example - Consider the following rhotrix and find its corresponding square root

$$A = \left\langle \begin{matrix} & 2 & \\ 1 & 9 & 1 \\ & 3 & \end{matrix} \right\rangle .$$

the square root of A is

$$B = \left\langle \begin{matrix} & 1/3 & \\ 1/6 & 3 & 1/6 \\ & 1/2 & \end{matrix} \right\rangle .$$

1.3.4 nth root of a rhotrix

Let $A = \left\langle \begin{matrix} & a_1 & \\ a_2 & h(A) & a_3 \\ & a_4 & \end{matrix} \right\rangle$ be a rhotrix with a positive heart. The nth power of A is the following

$$A^n = \left\langle \begin{matrix} & na_1h(A) & \\ n\,a_2h(A) & h(A)^n & na_3h(A) \\ & na_4h(A) & \end{matrix} \right\rangle$$

Since we can easily determine the nth power of a rhotrix, we can use it to evaluate the nth root of a given positive rhotrix.

It can easily be verified that

$$B = \left\langle \begin{matrix} & b_1 & \\ b_2 & h(B) & b_3 \\ & b_4 & \end{matrix} \right\rangle$$

is the nth root of A, where $h(B) = \sqrt[n]{h(A)}$ $b_i = a_i/n\sqrt{h(A)}$,i = 1,..,4. [1]

1.4 : TYPES OF RHOTRICES

1.4.1 SOME IMPORTANT DEFINITIONS

- **Singleton Rhotrix** :- A Rhotrix that has only one element (i.e. heart of Rhotrix) is called a singleton Rhotrix. A singleton rhotrix is represented as $\langle h(R) \rangle$, for example $\langle 5 \rangle$. In the above example of the Singleton rhotrix, there is only one element '5'.
- **Equal Rhotrix** :- The two Rhotrices are said to be equal if and only if the dimension of both Rhotrices is the same and also if their corresponding elements are equal. Let us consider two Rhotrices,

$$R_3 = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & c_1 & d_1 \\ & e_1 & \end{array} \right\rangle \text{ and } Q_3 = \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & c_2 & d_2 \\ & e_2 & \end{array} \right\rangle$$

Then R_3 and Q_3 are equal if $a_1=a_2$, $b_1=b_2$, $c_1=c_2$, $d_1=d_2$, $e_1=e_2$.

- **Orthogonal Rhotrix** :- An orthogonal Rhotrix is a rhotrix whose transpose is equal to the inverse of the rhotrix i.e. $R_n^{T/2} = (R_n)^{-1}$ or $(R_n)(R_n^{T/2}) = (R_n^{T/2})(R_n) = I$.
- **Diagonal Rhotrix** :- There two types of diagonal Rhotrices - Vertical Diagonal Rhotrix and Horizontal Diagonal Rhotrix . In the Vertical diagonal Rhotrix, all the elements except the vertical diagonal are zeros. For example,

$$R3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 2 & 0 \\ & 3 & \end{array} \right\rangle ; R7 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 2 & 0 & \\ 0 & 0 & 3 & 0 & 0 \\ & 0 & 4 & 0 & \\ & & 5 & & \end{array} \right\rangle$$

In the Horizontal diagonal Rhotrix, all the elements except the horizontal diagonal are zeros. For example,

$$R3 = \left\langle \begin{array}{ccc} & 0 & \\ 1 & 2 & 3 \\ & 0 & \end{array} \right\rangle ; R7 = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 1 & 2 & 3 & 4 & 5 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle$$

- **Upper Triangular Rhotrix** :- An upper triangular Rhotrix is a Rhotrix, whose all elements below the horizontal diagonal are zeros. For example,

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 3 & 4 \\ & 0 & \end{array} \right\rangle ; R_7 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & 3 & 4 & \\ 5 & 6 & 7 & 8 & 9 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle$$

- **Lower Triangular Rhotrix** :- A lower triangular Rhotrix is a Rhotrix, whose all elements above the horizontal diagonal are zeros. For example,

$$R_3 = \left\langle \begin{array}{ccc} & 0 & \\ 2 & 3 & 4 \\ & 5 & \end{array} \right\rangle ; \quad R_7 = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & \\ & & 9 & & \end{array} \right\rangle$$

- **Left Triangular Rhotrix** :- A left triangular Rhotrix is a Rhotrix, whose all elements on the left hand side of the vertical diagonal are zeros. For example,

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 2 & 3 \\ & 4 & \end{array} \right\rangle ; \quad R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 2 & 3 & \\ 0 & 0 & 4 & 5 & 6 \\ & 0 & 7 & 8 & \\ & & 9 & & \end{array} \right\rangle$$

- **Right Triangular Rhotrix** :- A right triangular Rhotrix is a Rhotrix, whose all elements on the right hand side of the vertical diagonal are zeros. For example,

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 3 & 0 \\ & 4 & \end{array} \right\rangle ; \quad R_7 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & 3 & 0 & \\ 4 & 5 & 6 & 0 & 0 \\ & 7 & 8 & 0 & \\ & & 9 & & \end{array} \right\rangle$$

- **Spectrum of Rhotrices** :- The set of Eigenvalues of a rhotrix R_n is known as the spectrum of R_n. [13]

1.4.2 EVEN DIMENSIONAL RHOTRIX (HEARTLESS RHOTRIX)

A Rhotrix $A = \{\left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle : a, b, d, e \in \mathbb{R}\}$

is called a real rhotrix of dimension two. This is a set of all two even-dimensional (heartless) rhotrices. We shall simply refer to these even - dimensional rhotrices as heartless rhotrices with an acronym hl-rhotrices. Accordingly, the cardinality of n-dimensional real hl-rhotrix is denoted as $|\hat{R}_n(\mathfrak{R})| = \frac{1}{2}(n^2 + 2n)$.where n $\in$ 2**N**. This implies that all hl-rhotrices are of even dimension. Therefore, all hl-rhotrices are rhotrices but all rhotrices are not hl-rhotrices.

Operations

- **Addition of hl-Rhotrices**

  Consider 2 Rhotrices $A = \left\langle \begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array} \right\rangle ; \; B = \left\langle \begin{array}{ccc} & b_{11} & \\ b_{21} & & b_{12} \\ & b_{22} & \end{array} \right\rangle .$

  We define addition (+) by

$$A + B = \left\langle \begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & b_{11} & \\ b_{21} & & b_{12} \\ & b_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a_{11} + b_{11} & \\ a_{21} + b_{21} & & a_{12} + b_{12} \\ & a_{22} + b_{22} & \end{array} \right\rangle$$

It can easily be shown that (A, +) is a commutative group with the zero and negative elements being:

$0 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & 0 & \end{array} \right\rangle$ and $-A = \left\langle \begin{array}{ccc} & -a_{11} & \\ -a_{21} & & a_{12} \\ & -a_{22} & \end{array} \right\rangle$, respectively.

- **Scalar Multiplication**

  The scalar multiplication is defined as follows: If $\lambda \in R$ is a scalar and A is a hl-rhotrix, then $\lambda A = \lambda \left\langle \begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \lambda a_{11} & \\ \lambda a_{21} & & \lambda a_{12} \\ & \lambda a_{22} & \end{array} \right\rangle.$

- **Multiplication of hl-Rhotrices**

  The multiplicative operation of hl-rhotrices will be the row-column multiplication. This method as proposed by Sani is naturally suitable for hl-rhotrices.

  Hence, we define multiplication as follows:

$$A \circ B = \left\langle \begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & b_{11} & \\ b_{21} & & b_{12} \\ & b_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a_{11}b_{11} + a_{12}b_{21} & \\ a_{21}b_{11} + a_{22}b_{21} & & a_{11}b_{12} + a_{12}b_{22} \\ & a_{21}b_{12} + a_{22}b_{22} & \end{array} \right\rangle$$

  In other words, the multiplication of higher hl-rhotrices is still according to Sani with the empty heart treated as null element or zero-valued element. Treating our rhotrices this way allows us to see the higher dimensional hl-rhotrices as coupled matrices with the lower dimensional squared matrix (minor matrix) coupled in the higher dimensional squared matrix (major matrix).

  It can easily be verified that the set of all hl-rhotrices with multiplicative operation defined this way is a non-commutative algebra.

- **Identity element**

  Consider an hl-rhotrix A of n-dimensional, if I is also an hl-rhotrix of n-dimensional such that: $A \circ I = A = I \circ A.$

  Then I is an identity element.

  For a 2-dimensional hl-rhotrix, the identity element is given as: $I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array} \right\rangle$

  A 2-dimensional identity hl-rhotrix is presented for illustratıon purposes. For higher hl-rhotrices, the identity element is reduced accordingly by allowing entries at the major diagonal to be unity except at the centre which is empty while other entries are zeros. We speak of a major diagonal because we are seeing our hl-rhotrices as coupled matrices.

- **Inverse of an even dimensional Rhotrix**

  The concept of identity element makes the inverse of an hl-rhotrix meaningful. If for any hl-rhotrix A, we can find another hl-rhotrix X such that :

  $$A \circ X = X \circ A = I,$$

  then X will be the inverse of A.

  For example, let

  $$A = \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle \quad \text{and} \quad X = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -b & & -d \\ & a & \end{array} \right\rangle$$

  Then,

  $$A^{-1} = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -b & & -d \\ & a & \end{array} \right\rangle$$

  provided ae 6= bd. Otherwise, A will be called a singular hl-rhotrix.It is to be noted that the majority of the heart-based rhotrices with a non-zero heart are non-singular or invertible rhotrices. Not all hl-rhotrices are invertible.

Properties and examples

Given any m-dimensional hl-rhotrix, $m \in 2\mathbf{N}$ and $m \geq 4$. The following hold:

- The heart will be missing from the major matrix if the dimension (D_m) lies in the set below: $\{D_{4+4n}: n \in \mathbf{N}\}$.
- The major matrix is called a surrogate major matrix if the dimension D_m of a hl-rhotrix lies in the set below: $\{D_{4+4n} : n \in \mathbf{N}\}$.
- The heart will be missing from the minor matrix if the dimension (D_m) lies in the set below: $\{D_{6+4n}: n \in \mathbf{N}\}$.
- The minor matrix is called a surrogate minor matrix if the dimension D_m of a hl-rhotrix lies in the set below: $\{D_{6+4n} : n \in \mathbf{N}\}$.

The set A of all hl-rhotrices form a vector space which is spanned by the following vector (rhotrices) $I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 0 & \end{array} \right\rangle; J = \left\langle \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ & 0 & \end{array} \right\rangle; K = \left\langle \begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & 0 & \end{array} \right\rangle; L = \left\langle \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & 1 & \end{array} \right\rangle.$

These vectors are linearly independent, therefore, the set S = {I, J, K, L} forms a basis for a 2-dimensional hl-rhotrix (R_2). Any 2-dimensional hl-rhotrix (A), $A = \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle$ can be written as a linear combination of the rhotrices in S so that,

$$A = aI + bK + dK + eL.$$

Now, we can look through some examples of higher dimensional heartless Rhotrices with their coupled matrix alone.

If R_n is an hl-rhotrix, then R_c n will denote the corresponding coupled matrix.

- A hl-rhotrix of dimension four (R4) is given by:

$$R_4 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle .$$

Then its corresponding coupled matrix will be presented below:

$$R_4^c = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{12} & \\ a_{21} & & & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix} .$$

Because of the absence of a heart, we have a surrogate 3 × 3 matrix (major matrix) coupled with a 2 × 2 matrix (minor matrix). Representation of heart-based rhotrices(h-rhotrices) into coupled matrices was introduced by Sani. In this case of R4, the missing heart is in the surrogate matrix.

- A hl-rhotrix of dimension six (R6) is given by:

$$R_6 = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ a_{41} & c_{31} & a_{32} & & a_{23} & c_{13} & a_{14} \\ & a_{42} & c_{32} & a_{33} & c_{23} & a_{24} & \\ & & a_{43} & c_{33} & a_{34} & & \\ & & & a_{44} & & & \end{array} \right\rangle .$$

Then its corresponding coupled matrix is:

$$R_6^c = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} & & a_{14} \\ & c_{11} & & c_{12} & & c_{13} & \\ a_{21} & & a_{22} & & a_{23} & & a_{24} \\ & c_{21} & & & & c_{23} & \\ a_{31} & & a_{32} & & a_{33} & & a_{34} \\ & c_{31} & & c_{32} & & c_{33} & \\ a_{41} & & a_{42} & & a_{43} & & a_{44} \end{bmatrix} .$$

In this case of R_2, we have a surrogate 3 × 3 minor matrix and the 4 × 4 major matrix. The missing heart is in the surrogate 3 × 3 matrix.

Comparison between odd and even dimensional rhotrices [16].

The similarities and differences between h-rhotrices and hl-rhotrices are presented below:

	h-rhotrix	hl-rhotrix
1	Equal row and column	Equal row and column
2	The heart exists	The heart does not exist
3	Mostly invertible, provided the heart is not zero	Not all are invertible
4	It gives two squared-coupled matrix	It gives a squared-surrogate coupled matrix
5	Odd dimensional	Even dimensional
6	$\lvert R_n\rvert = \frac{1}{2}(n^2+1), n \in 2\mathbb{Z}^+ + 1$	$\lvert R_n\rvert = \frac{1}{2}(n^2+2n), n \in 2\mathbb{N}$
7	$\lvert R_n\rvert = m$ where n, m are both odd	$\lvert R_{n-1}\rvert = m-1$ (both even)

1.4.3 CLASSIFICATION OF RHOTRICES OVER NUMBER FIELDS

Natural Rhotrix:- A rhotrix is called a natural rhotrix if all its entries belong to the set of natural numbers. For example, $\hat{R}_3(\aleph) = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \aleph \right\}$

is the set of all three dimensional natural Rhotrices. Furthermore, the set of all natural rhotrices of the same dimension n, together with the operations of addition (+) , scalar multiplication (α) and multiplication (o) forms the natural rhotrix space.

Integer Rhotrix:- A rhotrix set is called an integer rhotrix set if all its entries belong to the set Z of integer numbers. Furthermore, the set of all integer rhotrices of the same dimension n, together with the operations of addition(+), scalar multiplication (α) and multiplication (o) forms the integer rhotrix space.

Rational Rhotrix :- A rhotrix set is called a rational rhotrix set if all its entries belong to the set Q of rational numbers. For example, $\hat{R}_3(Q^c) = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in Q^c \right\}$

is the set of all three dimensional rational rhotrices. Furthermore, the set of all rational rhotrices of the same dimension n, together with the operations of addition (+) , scalar multiplication (α) and multiplication (o) forms the rational rhotrix space, denoted by the pair $\left(\hat{R}_n(Q), +, \circ \right)$

Irrational Rhotrix :- A rhotrix set is called an irrational rhotrix set if all its entries belong to the set Q^c of irrational numbers. Furthermore, the set of all irrational rhotrices of the same dimension n, together with the operations of addition (+) , scalar multiplication (α) and multiplication (o) do not form a space of irrational rhotrices. Because, the set of irrational rhotrices of the same dimension n is not closed with respect to rhotrix multiplication (o) . Thus, we refer to this space as non-irrational rhotrix space.

Real Rhotrix :- A rhotrix set is called a real rhotrix set if all its entries belong to the set of real Numbers. For example,

$$\hat{R}_3(\Re) = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a,b,c,d,e \in \Re \right\}$$

is the set of all three dimensional real rhotrices. Furthermore, the set of all real rhotrices of the same dimension n, together with the operations of addition (+) , scalar multiplication (α) and multiplication (o) forms the real rhotrix space, denoted by the pair $\left(\hat{R}_n(\Re),+,\circ\right)$

Complex Rhotrix :- A rhotrix set is called a complex rhotrix set if all its entries belong to the set C of complex numbers.

For example,

$$\hat{R}_3(C) = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a,b,c,d,e \in C \right\}$$

is the set of all three dimensional rational rhotrices. Furthermore, the set of all complex rhotrices of the same dimension n, together with the operations of addition (+) , scalar multiplication (α) and multiplication (o) forms the complex rhotrix space, denoted by the pair $\left(\hat{R}_n(C),+,\circ\right)$

The rhotrix sets we have categorized over number fields satisfy the chain of rhotrix sets inclusions, given by $\hat{R}_n(\aleph) \subset \hat{R}_n(Z) \subset \hat{R}_n(Q) \subset \hat{R}_n(\Re) \subset \hat{R}_n(C)$

Analogously, the rhotrix spaces we have categorized over number fields, satisfy the chain of rhotrix spaces inclusions, given by

$$\left(\hat{R}_n(\aleph),+,\circ\right) \subset \left(\hat{R}_n(Z),+,\circ\right) \subset \left(\hat{R}_n(Q),+,\circ\right) \subset \left(\hat{R}_n(\Re),+,\circ\right) \subset \left(\hat{R}_n(C),+,\circ\right)$$

Chapter 2

COMPARISON OF MATRIX THEORY AND RHOTRIX THEORY

2.1 COMPARISON OF MATRICES AND RHOTRICES

Algebraic Properties :

Matrices and rhotrices show similar behaviour when it comes to their algebraic properties. Some of the properties are listed below [4] :

$$A + 0 = 0 = A = A$$

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$\alpha(A + B) = AB + AC$$

$$A(B + C) = AB + AC$$

$$A(BC) = (AB)C$$

$$AI = A = IA$$

Here, when applied to rhotrices, 0 implies the zero rhotrix of corresponding dimension, multiplication operation used is the row-column based multiplication, and I is the identity rhotrix of the appropriate dimension corresponding to row-column multiplication.

Elementary Row Operations :

Three elementary row operations can be performed on a matrix representing a linear system of equations without changing the nature or values of the system of linear equations:

Interchanging rows

Multiplication of a row by a scalar to give a new row

Adding a row with other rows

The same is applicable for columns as well. These elementary operations can be applied to rhotrices representing two linear systems of equations without any change in their solutions. It is important to note that these operations can only be performed using two major rows or two minor rows, and not between one major row and one minor row.

Rank of Matrices and Rhotrices :

The rank of a matrix A, denoted by rank(A), is defined as the number of non-zero rows in its row-reduced echelon form. Similarly, rank of an n-dimensional rhotrix can be defined using the major and minor matrices in its coupled matrix form with dimensions $\frac{n+1}{2}$ and $\frac{n-1}{2}$ respectively. This approach opens possibilities for many properties of matrices related to their ranks to be extended to rhotrices.

Let Rn = $\langle a_{ij}, c_{kl} \rangle$ be the coupled matrix form of an n-dimensional rhotrix R. The rank of R can be defined as

$$\text{rank}(R) = \text{rank}(a_{ij}) + \text{rank}(c_{kl})$$

Using this, we can obtain the following results for two n-dimensional rhotrices R and S:

$$\text{rank}(R) \leq n$$

$$\text{rank}(R+S) \leq \text{rank}(R) + \text{rank}(S)$$

$$\text{rank}(R) + \text{rank}(S) - n \leq \text{rank}(RS)$$

$$\text{rank}(RS) \leq \min\{\text{rank}(R), \text{rank}(S)\}$$

Filled Coupled Matrix :

For any odd integer n, the n-dimensional square matrix a_{ij} is called a filled coupled matrix if a_{ij}= 0 when i+j ϵ $2Z^+$+1. These entries are called null entries of the filled coupled matrix. A filled coupled matrix can be defined corresponding to every rhotrix, and the dimension of the filled coupled matrix will be the same as the dimension of the rhotrix.

Gaussian elimination, followed by back substitution, can be performed on the filled coupled matrix to solve the systems of linear equations [2].

2.2 VECTOR SPACES [2]

Representation of vectors: The representation of vectors in matrix form and rhotrix form are vastly different from each other.

$$\left\langle \begin{array}{c} 0 \\ 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{t1} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{tt-2} \quad 0 \quad 0 \quad 0 \quad 0 \\ a_{tt-1} \quad 0 \quad 0 \end{array} \right\rangle$$ is an n-dimensional rhotrix row vector, and

$$\left\langle \begin{array}{c} a_{11} \\ a_{21} \quad 0 \quad 0 \\ a_{31} \quad 0 \quad 0 \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{t1} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \\ 0 \end{array} \right\rangle$$ is an n-dimensional rhotrix column vector, where $t = \frac{n+1}{2}$.

While matrix row vectors or column vectors have unique representations, a rhotrix row vector or column vector can be represented in t different ways. For example,

(1)

$$\left\langle \begin{array}{c} x_1 \\ 0 \quad 0 \quad x_2 \\ 0 \quad 0 \quad 0 \quad 0 \quad x_3 \\ 0 \quad 0 \quad 0 \\ 0 \end{array} \right\rangle$$

(2)

$$\left\langle \begin{array}{c} 0 \\ x_1 \quad 0 \quad 0 \\ 0 \quad 0 \quad x_2 \quad 0 \quad 0 \\ 0 \quad 0 \quad x_3 \\ 0 \end{array} \right\rangle$$

(3)

$$\left\langle \begin{array}{c} 0 \\ 0 \quad 0 \quad 0 \\ x_1 \quad 0 \quad 0 \quad 0 \quad 0 \\ x_2 \quad 0 \quad 0 \\ x_3 \end{array} \right\rangle$$

the rhotrices (1), (2), and (3) are different representations of the same five-dimensional rhotrix row vector whereas a three-dimensional row vector is uniquely represented as (x_1, x_2, x_3).

Definition of a rhotrix vector space:

A rhotrix vector space $\langle V \rangle$ over the set of real numbers is a non-empty set of rhotrix vectors with two operations vector addition and scalar multiplication, obeying the ten axioms of vector space in linear algebra. A non-empty set of matrix vectors with the operations vector addition and scalar multiplication obeying the ten axioms of vector space is called a matrix vector space.

Properties : Let a, b, and c be three n-dimensional vectors with the same representation. Let α and β be two real numbers. The three vectors obey the following properties whether they are matrix vectors or rhotrix vectors.

1) $a + \mathbf{0} = a$, where **0** is the corresponding zero vector
2) $0 \cdot a = 0$, where 0 is the real number
3) $a + b = b + a$
4) $(a + b) + c = a + (b + c)$
5) $(-1) \cdot a = -a$
6) $a + (-a) = \mathbf{0}$
7) $1 \cdot a = a$
8) $\alpha \cdot (a + b) = \alpha \cdot a + \alpha \cdot b$
9) $(\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a$
10) $(\alpha\beta) \cdot a = \alpha \cdot (\beta \cdot a)$

Linear Dependence :

The vector space $\langle V \rangle$ is said to be linearly dependent or ,simply, dependent if there exist a set of scalars(not all of which are zero) in the set of real numbers and a set of vectors in $\langle V \rangle$ such that their linear combination equals to zero. [12]

If the linear combination gives zero only when all the scalars have the same value that is equal to zero, then $\langle V \rangle$ is called linearly independent or, simply, independent.

Linear Mapping and Dimension of Vector Space :

In linear algebra, the concept of representing linear mappings as matrices is of great importance. Since the dimension of a rhotrix is always odd, it follows that, in representing a linear map T on a vector space $\langle V \rangle$, the dimension of $\langle V \rangle$ has to necessarily be odd. This restriction does not occur in the case of matrix linear maps. The rhotrix representation of a linear mapping over an odd-dimensional vector space $\langle V \rangle$ can be transformed to a matrix linear map where the matrix will be the filled coupled matrix of the rhotrix.

Cramer's Rule :

Cramer's rule is a method for finding solutions to systems of linear equations with an equal number of equations and unknowns. In linear algebra, Cramer's rule uses the matrix representation of the linear mapping to solve the system of linear equations. This can be extended to rhotrix linear mappings to find solutions to systems of linear equations represented by the rhotrix. [11]

Cayley-Hamilton Theorem :

The Cayley-Hamilton theorem is a cornerstone of linear algebra, establishing a profound relationship between a matrix and its characteristic polynomial. This theorem has far-reaching implications in various fields, including engineering, physics, and computer science.

The Cayley-Hamilton theorem for matrices states that every square matrix satisfies its own characteristic equation. This means that if we substitute the matrix A into its characteristic polynomial, the result is always the zero matrix.

The concepts of eigenvalues and eigenvectors have been established in the literature regarding rhotrix theory. Eigenvalues of a rhotrix are the solutions to the characteristic polynomial obtained for that rhotrix. It is also to be noted that the Cayley-Hamilton theorem can be extended to rhotrices. The equivalence of the result has been established by Abdulhadi Aminu in 2012 in his paper titled “Cayley-Hamilton Theorem in Rhotrices”. The theorem says that every rhotrix satisfies its own characteristic equation.

Chapter 3

SOME CONCEPTS OF RHOTRICES

3.1 RHOTRIX EXPONENT RULE

THEOREM

Let *R* be a rhotrix. Then for any integer values of m,

$$R^m = (h(R))^{m-1}\left\langle \begin{array}{ccc} & ma & \\ mb & h(R) & md \\ & me & \end{array} \right\rangle$$

(Let this be equation 1).

In particular,

(i) R^0 is the identity of *R* and

(ii) R^{-1} is the inverse of *R*.

PROOF

We shall establish this theorem using the principle of mathematical induction. First, we consider the case for positive integer values of m. The result is certainly true for m=1.Now suppose it is true for m=k. Then,we have,

$$R^k = (h(R))^{k-1}\left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle$$

$$R^{k+1} = R^K \circ R^1 = (h(R))^{k-1}\left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

$$\Rightarrow R^{k+1} = (h(R))^{k}\left\langle \begin{array}{ccc} & (k+1)a & \\ (k+1)b & h(R) & (k+1)d \\ & (k+1)e & \end{array} \right\rangle$$

Thus, the theorem holds for the power (k+1)and so it is true for any positive integer. Next, if m is a negative integer, write m =-k , so that k is a positive integer. Then by the definition of quotient rhotrix , we have $R^m = R^{-k} = \frac{I}{R^k} = \frac{I}{(h(R))^{k-1}\left\langle \begin{matrix} & ka & \\ kb & h(R) & kd \\ & ke & \end{matrix} \right\rangle} = (h(R))^{-(k-1)}\left\langle \begin{matrix} & ka & \\ kb & h(R) & kd \\ & ke & \end{matrix} \right\rangle^{-1}$

$$\Rightarrow R^m = R^{-k} = (h(R))^{-(k-1)}.\frac{-1}{(h(R))^2}\left\langle \begin{matrix} & ka & \\ kb & -h(R) & kd \\ & ke & \end{matrix} \right\rangle = (h(R))^{-k-1}\left\langle \begin{matrix} & -ka & \\ -kb & h(R) & -kd \\ & -ke & \end{matrix} \right\rangle$$

Provided, h(R)≠0. Therefore, the theorem holds for all negative values of m. Finally, if m=0, we have:

$$\Rightarrow R^0 = \frac{1}{(h(R))}\left\langle \begin{matrix} & (k-k)a & \\ (k-k)b & h(R) & (k-k)d \\ & (k-k)e & \end{matrix} \right\rangle$$

$$\Rightarrow R^0 = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$$

Thus, the theorem holds for the power m = 0.

Hence, the theorem is true for all integer values of m.

As a corollary to the theorem on rhotrix exponent rule, we have the following:

1) If m = 0 and m = -1 in the previous theorem . Then

(i) R^0 is the identity of rhotrix R and

(ii) R^{-1} is the inverse of rhotrix R respectively.

Proof. (i) If m = 0 in previous theorem, we have:

$$R^0 = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$$

Therefore, $R \circ R^0 = R^0 \circ R = R$

Hence, R^0 is the identity of rhotrix *R*.

(ii) If m = -1 in the previous theorem , we have:

$$R^{-1} = (h(R))^{-2}\left\langle \begin{matrix} & -a & \\ -b & h(R) & -d \\ & -e & \end{matrix} \right\rangle = \frac{-1}{(h(R))^2}\left\langle \begin{matrix} & a & \\ b & -h(R) & d \\ & e & \end{matrix} \right\rangle$$

Thus, R^{-1} is the inverse of rhotrix R.[3]

The following corollary is obvious:

Corollary

If R is a unit heart rhotrix then we have for any integer values of m,

$$R^m = \left\langle \begin{array}{ccc} & ma & \\ mb & 1 & md \\ & me & \end{array} \right\rangle$$

PROOF

Substituting h(R)=1 in equation (1), then the result follows.

Properties of the rhotrix exponent rule:

(a) $R^m \circ R^n = R^{m+n}$

(b) $\dfrac{R^m}{R^n} = R^{m-n}, provided, h(R) \neq 0$

(c) $\left(R^m\right)^{1/n} = R^{\frac{m}{n}}$

(d) $\left(R^m\right)^n = R^{mn}$

(e) $\left(kR\right)^m = k^m R^m$ (Where k is a scalar)

(f) $R^0 = I$ (Where I is the identity of R)

(g) $R^{-1} = \dfrac{I}{R} = I \circ R^{-1}, provided, h(R) \neq 0$

(h) $R^m = 0$ (or zero rhotrix), provided $h(R) = 0$ and m≥2

3.2 : RHOTRIX LINEAR TRANSFORMATION

Rank of a Rhotrix :

Let $R_n = \langle a_{ij}, c_{kl} \rangle$, the entries $a_{rr}\ (1 \leq r \leq t)$ and $c_{ss}\ (1 \leq s \leq t-1)$ in the main diagonal of the major and minor matrices of R respectively, formed the main diagonal of R. If all the entries to the left (right) of the main diagonal in are zeros, is called a right (left) triangular rhotrix. The following lemma follows trivially.

Lemma :

Let , $R_n = \langle a_{ij}, c_{kl} \rangle$ is a left (right) triangular rhotrix if and only if (a_{ij}) and (c_{kl}) are lower (upper) triangular matrices.

Proof :

This follows when the rhotrix n is being rotated through 45° in an anticlockwise direction. R

In the light of this lemma, any n-dimensional rhotrix can be reduced to a right triangular rhotrix by reducing its major and minor matrix to echelon form using elementary row operations. Recall that, the rank of a matrix (A) denoted by rank (A)is the number of non-zero row(s) in its reduced row echelon form. If , $R_n = \langle a_{ij}, c_{kl} \rangle$ we define rank of R denoted by Rank (A) as:

$$\text{rank}(R) = \text{rank}(a_{ij}) + \text{rank}(c_{kl}) \tag{3}$$

Example : -

Let ,

$$A = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 2 & -2 & \\ 1 & -1 & 3 & 1 & 2 \\ & -2 & 1 & 1 & \\ & & 2 & & \end{array} \right\rangle .$$

Then, the filled coupled matrix of A is given by

$$m(A) = \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 2 \end{pmatrix} .$$

Now reducing m(A) to reduce row echelon form (rref) , we obtain

$$rref(m(A)) = \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is a coupled of (2 x 2) and (3 x 3) matrices, i.e.

$$A(\text{say}) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } B(\text{say}) = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ respectively.

Notice that,

$$\text{rank}(A) + \text{rank}(B)$$
$$= 2 + 2 = 4 = \text{rank}(rref(m(A))).$$

Hence, $= \text{rank}(A) = 4$.

Rhotrix Linear Transformation : One of the most important concepts in linear algebra is the concept of representation of linear mappings as matrices. If V and W are vector spaces of dimension n and m respectively, then any linear mapping T from V to W can be represented by a matrix. The matrix representation of T is called the matrix of T denoted by m(T) . Recall that, if F is a field, then any vector space V of finite dimension n over V of finite dimension n over F is isomorphic to F^n. Therefore, any n x n matrix over F can be considered as a linear operator on the vector space F^n in the fixed standard basis. Following these ideas, we study in this section, a rhotrix as a linear operator on the vector space F^n.Since the dimension of the rhotrix is always odd ,it follows that ,in representing a linear map T on a vector space V by a rhotrix ,the dimension of V is necessarily odd . Therefore, throughout what follows, we shall consider only odd dimensional vector spaces.For any n belongs to $2Z^+ + 1$ And F be an arbitrary field, we find the coupled F^t , F^{t-1},of F^t

$$F^t = \{(\alpha_1, \alpha_2, \cdots, \alpha_t) | \alpha_1, \cdots, \alpha_t \in F\} \text{ and}$$
$$F^{t-1} = \{(\beta_1, \beta_2, \cdots, \beta_t) | \beta_1, \beta_2, \cdots, \beta_{t-1} \in F^{t-1}\} \text{ by}$$

$$(F^t, F^{t-1}) = \{(\alpha_1, \alpha_2, \cdots, \alpha_t, \beta_1, \beta_2, \cdots, \beta_{t-1}) : \alpha_1, \alpha_2, \cdots, \alpha_t, \beta_1, \beta_2, \cdots, \beta_{t-1} \in F^t\}.$$

It is clear that (F^t, F^{t-1}) coincides with F^n and so, if $n \in 2Z^+ + 1$,any n-dimensional vector spaces V1 and V2 is of dimensions $\frac{n+1}{2}$ and $\frac{n+1}{2} - 1$ respectively. Less obviously, it can be seen that not every linear map of T of F^n can be represented by a rhotrix in the standard basis. For instance, the map
$T : F^3 \to F^3$

Defined by $T(x, y, z) = (x - y, x + z, y + z)$ is a linear mapping on F^3 which cannot be represented by a rhotrix in the standard basis. The following theorem characterizes when a linear map T on F^n can be represented by a rhotrix .

Theorem :

Let $n \in 2Z^+ + 1$, and F be a field. Then, a linear map $T : F^n \to F^n$ can be represented by a rhotrix with respect to the standard basis if and only if T is defined as

$$T(x_1, y_1, x_2, y_2, \cdots, y_{t-1}, x_t)$$
$$= (\alpha_1(x_1, x_2, \cdots, x_t), \beta_1(y_1, y_2, \cdots, y_{t-1}), \alpha_2(x_1, x_2, \cdots, x_t), \beta_2(y_1, y_2, \cdots, y_{t-1}), \cdots, \beta_{t-1}(y_1, y_2, \cdots, y_{t-1}), \alpha_t(x_1, x_2, \cdots, x_t)),$$

where $t = \frac{n+1}{2}, \alpha_1, \cdots, \alpha_t$ and $\beta_1, \cdots, \beta_{t-1}$ are any linear map on F^t and F^{t-1} respectively.

Proof:

Suppose $T: F^n \to F^n$ is defined by

$$\begin{aligned} &T(x_1, y_1, x_2, y_2, \cdots, y_{t-1}, x_t) \\ &= (\alpha_1(x_1, x_2, \cdots, x_t), \beta_1(y_1, y_2, \cdots, y_{t-1}), \\ &\quad \alpha_2(x_1, x_2, \cdots, x_t), \beta_2(y_1, y_2, \cdots, y_{t-1}), \cdots, \\ &\quad \beta_{t-1}(y_1, y_2, \cdots, y_{t-1}), \alpha_t(x_1, x_2, \cdots, x_t)), \end{aligned}$$

where, $t = \frac{n+1}{2}, \alpha_1, \cdots, \alpha_t$ and $\beta_1, \cdots, \beta_{t-1}$ are any linear map on F^t and F^{t-1} respectively, and consider the standard basis

$\{(1,0,\cdots,0),(0,1,0,\cdots,0),\cdots,(0,0,\cdots,1)\}$.

Note that for , $1 \le i \le t$ and $1 \le j \le t-1$. Since α_i, β_j are linear maps

Thus ,

$$\left.\begin{aligned} T(1,0,\cdots,0) &= [\alpha_1(1,0,\cdots,0), 0, \cdots, \alpha_t(1,0,\cdots,0)] \\ T(1,0,\cdots,0) &= [0, \beta_1(1,0,\cdots,0), \cdots, \beta_{t-1}(1,0,\cdots,0)] \\ &\vdots \\ T(0,\cdots,0,1) &= [0, \beta_1(0,\cdots,0,1), \cdots, \beta_{t-1}(0,\cdots,0), 1] \\ T(0,\cdots,0,1) &= [\alpha_1(0,\cdots,0,1), 0, \cdots, \alpha_t(0,0,\cdots,0,1)] \end{aligned}\right\} \quad (5)$$

Let $\alpha_{ij} = \alpha_j\left(0,\cdots,\underset{i^{\text{th}}-\text{position}}{1},\cdots,0\right)$ for $(1 \le i, j \le t)$ and $\beta_{kl} = \beta_l\left(0,\cdots,\underset{j^{\text{th}}-\text{position}}{1},\cdots,0\right)$ for $(1 \le k, l \le t-1)$. Then from (5), we have the matrix of T is

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & \cdots & \alpha_{1t-1} & 0 & \alpha_{1t} \\ 0 & \beta_{11} & 0 & \cdots & 0 & \beta_{1t-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \beta_{t-1t} & 0 & \cdots & 0 & \beta_{t-1t-1} & 0 \\ \alpha_{t1} & 0 & \alpha_{t2} & \cdots & \alpha_{tt-1} & 0 & \alpha_{tt} \end{pmatrix}. \quad (6)$$

This is filled coupled matrix from which we obtain the rhotrix representation of T as $\langle \alpha_{ij}, \beta_{kl} \rangle$

Conversely:

Suppose $T: F^n \to F^n$ has a rhotrix representation $\langle \alpha_{ij}, \beta_{kl} \rangle$ in the standard basis. Then, the corresponding matrix representation of T is the filled coupled given in (6) above. Thus, we obtain the system

$$\left.\begin{array}{l} T(1,0,\cdots,0)=(\alpha_{11},0,\alpha_{12},\cdots,\alpha_{1t-1},0,\alpha_{1t}) \\ T(1,0,\cdots,0)=(0,\beta_{1t-1},0,\cdots,\beta_{1t-1},0) \\ \vdots \\ T(0,\cdots,0,1)=(0,\beta_{t-1t},0,\cdots,\beta_{t-1t-1},0) \\ T(0,\cdots,0,1)=(\alpha_{t1},0,\alpha_{t2},\cdots,\alpha_{tt-1},0,\alpha_{tt}) \end{array}\right\} \quad (7)$$

From this system ,it follows that for each $(x_1,y_1,x_2,y_2,\cdots,y_{t-1},x_t)\in F^n$ we have the linear transformation T defined by

$$\begin{aligned} &T(x_1,y_1,x_2,y_2,\cdots,y_{t-1},x_t) \\ &=(\alpha_1(x_1,x_2,\cdots,x_t),\beta_1(y_1,y_2,\cdots,y_{t-1}), \\ &\quad \alpha_2(x_1,x_2,\cdots,x_t),\beta_2(y_1,y_2,\cdots,y_{t-1}),\cdots, \\ &\quad \beta_{t-1}(y_1,y_2,\cdots,y_{t-1}),\alpha_t(x_1,x_2,\cdots,x_t)), \end{aligned}$$

where , $t=\frac{n+1}{2},\alpha_1,\cdots,\alpha_t$ and $\beta_1,\cdots,\beta_{t-1}$ are any linear map on F^t with $\alpha_j\left(0,\cdots,\underset{i^{th}-\text{position}}{1},\cdots,0\right)=\alpha_{ij}$ for $(1\le i,j\le t)$ and $\beta_l\left(0,\cdots,\underset{j^{th}-\text{position}}{1},\cdots,0\right)=\beta_{kl}$ for $(1\le k,l\le t-1)$.

Example :-

Consider the linear mapping $T:\Re\to\Re$ defined by $T(x,y,z)=(2x-z,4y,x-3z)$ .To find the rhotrix of T relative to the standard basis .We proceed by finding the matrices of T .

Thus ,

$$T(1,0,0)=(2,0,1)$$
$$T(0,1,0)=(0,4,0)$$
$$T(0,0,1)=(-1,0,-3)$$

Therefore , by definition of the matrix of T with respect to the standard basis , we have

$$m(T)=\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{pmatrix},$$

which is the filled coupled matrix from which we obtain the rhotrix of T in R3,

$$r(T)=\left\langle \begin{array}{ccc} & 2 & \\ -1 & 4 & 1 \\ & -3 & \end{array} \right\rangle.$$

Now starting with the rhotrix $r(T) = \left\langle \begin{matrix} & 2 & \\ -1 & 4 & 1 \\ & -3 & \end{matrix} \right\rangle$ the filled coupled matrix of $r(T)$ is $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{pmatrix}$.

And so, defining $T : R_3 \to R_3$

$$T(1,0,0) = 2(1,0,0) + 0(0,1,0) + 1(0,0,1)$$
$$T(0,1,0) = 0(1,0,0) + 4(0,1,0) + 0(0,0,1)$$
$$T(0,0,1) = -1(1,0,0) + 0(0,1,0) - 3(0,0,1)$$

Thus, if $(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.
Therefore,

$$\begin{aligned} T(x,y,z) &= xT(1,0,0) + yT(0,1,0) + zT(0,0,1) \\ &= x(2,0,1) + y(0,4,0) + z(-1,0,-3) \\ &= (2x - z, 4y, x - 3z) \end{aligned}$$

[4].

3.3 SOME CONCEPTS OF RHOTRIX TYPE A SEMIGROUPS

R is any rhotrix while R_n is an n-dimensional rhotrix.

Definition : Suppose $R_n = \langle a_{ij}, c_{lk} \rangle$ is an n-dimensional rhotrix, then the determinant of R_n is given by, $\det(R_n) = \det(A_t) \det(C_{t-1})$ where A_t and C_{t-1} are two square matrices of dimension $(t \times t)$ and $(t-1) \times (t-1)$ respectively which make up the rhotrix R_n with $t = (n+1)/2$ and $n \in 2Z^+ + 1$.

Remark : A rhotrix R_n is said to be invertible or non-singular if the determinant is non-zero. That is, R_n is invertible if $\det(R_n) \neq 0$.

Theorem : For any rhotrix $R \neq 0$, $R^2 = 0$ if and only if $h(R) = 0$ where 0 is the zero rhotrix.

Theorem :

Let $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ be any rhotrix of size 3, then for any integer m,

$$R^m = (h(R))^{m-1} \left\langle \begin{matrix} & ma & \\ mb & h(R) & md \\ & me & \end{matrix} \right\rangle.$$

In particular, R^0 and R^{-1} are the identity and inverse of R respectively, provided $h(R)$ is non-zero.

Proposition : Let A, B and C be three rhotrices of the same size with entries in $\mathbb{R}$, then the system of linear equations resulting from $A \circ B = C$ has

i) a unique solution if and only if $h(A) \neq 0$ and $h(C) \neq 0$.

ii) an infinite solution if and only if $h(A) = h(C) = 0$.

iii) no solution if and only if $h(A) = 0$ and $h(C) \neq 0$.

Theorem : The rhotrix semigroup $(R_n(\text{F}), \circ)$ is embedded in the matrix semigroup $(M_n(\text{F}), .)$

Remark : Given the map $\theta : R_n(\text{F}) \to M_n(\text{F})$, the image set $\langle R_n(\text{F})\rangle\theta$ is a subsemigroup of $M_n(\text{F})$ consisting of all filled coupled $n \times n$ matrices. Using the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$, the following results was obtained.

Theorem : Suppose $A, B \in R_n^*(\text{F})$,then

i) $A \mathcal{L} B$ if and only if $im(A) = im(B)$.

ii) $A \mathcal{R} B$ if and only if $\ker(A) = \ker(B)$.

iii) $A \mathcal{H} B$ if and only if $im(A) = im(B)$ and $\ker(A) = \ker(B)$.

Lemma :

Let S be a semigroup and e be an idempotent in S. Then the following are equivalent in S

i) $a \ \mathcal{R}^* \ e$

ii) $ea = a$ and for all $x, y \in S^1$, $xa = ya \Rightarrow xe = ye$.

Definition : The relation σ on a type A semigroup S is defined by the rule that $(a, b) \in \sigma$ if and only if $ae = be$ for some $e \in E(S)$. It is known that σ is the minimum cancellative congruence on S. It is important to note that σ can also be written as $a \sigma b$ if and only if $fa = fb$ for some $f \in E(S)$.

RHOTRIX TYPE A SEMIGROUP

This section focuses on the construction of a rhotrix type A semigroup and the properties embedded in the semigroup constructed.

Now let $R_n(\text{F})$ be a set of all rhotrices of size n with entries from an arbitrary field F. For any $A_n, B_n \in R_n(\text{F})$, define a binary operation $\circ$ on $R_n(\text{F})$ by the rule:

$$A_n \circ B_n = \langle a_{i_1 j_1}, c_{l_i k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \langle \sum_{i_2 j_1 = 1}^{t} \left(a_{i_1 j_1} b_{i_2 j_2}\right), \sum_{l_2 k_1 = 1}^{t-1} \left(c_{l_i k_1} d_{l_2 k_2}\right) \rangle, t = \frac{n+1}{2},$$

where A_n and B_n denote n-dimensional rhotrices.

Theorem : $S = (R_n(F), \circ)$ is a semigroup.

Proof. Let $A_n, B_n A \in S$, we have that $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$, so that $A_n \circ B_n \in S$, since $\det(A_n \circ B_n) = \det(A_n) \times \det(B_n) \neq 0$. It follows that S is closed under the binary operation.

Next is to show that S is associative.

Suppose $A_n = \langle a_{i_1 j_1}, c_{l_i k_1} \rangle,\ B_n = \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle,\ C_n = \langle u_{i_3 j_3}, v_{l_3 k_3} \rangle,$

then we have that

$$\begin{aligned} A_n \circ (B_n \circ C_n) &= \langle a_{i_1 j_1}, c_{l_i k_1} \rangle \circ (\langle b_{i_2 j_2}, d_{l_2 k_2} \rangle \circ \langle u_{i_3 j_3}, v_{l_3 k_3} \rangle) \\ &= \langle a_{i_1 j_1}, c_{l_i k_1} \rangle \circ (\langle \textstyle\sum_{i_3 j_2=1}^{t} (b_{i_2 j_2} u_{i_3 j_3}), \sum_{l_3 k_2=1}^{t-1} (d_{l_2 k_2} v_{l_3 k_3}) \rangle) \\ &= \langle \textstyle\sum_{i_2 j_1=1}^{t} a_{i_1 j_1} [\sum_{i_3 j_2=1}^{t} (b_{i_2 j_2} u_{i_3 j_3})],\ \sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1} [\sum_{l_3 k_2=1}^{t-1} (d_{l_2 k_2} v_{l_3 k_3})] \rangle \\ &= \langle \textstyle\sum_{i_2 j_1=1}^{t} \sum_{i_3 j_2=1}^{t} a_{i_1 j_1} (b_{i_2 j_2} u_{i_3 j_3}),\ \sum_{l_2 k_1=1}^{t-1} \sum_{l_3 k_2=1}^{t-1} c_{l_1 k_1} (d_{l_2 k_2} v_{l_3 k_3}) \rangle \end{aligned}$$

Consequently, $A_n \circ (B_n \circ C_n) = (A_n \circ B_n) \circ C_n$.

Therefore S is a semigroup.

Lemma- Let $\langle a_{ij}, c_{lk} \rangle, \langle b_{ij}, d_{lk} \rangle \in S = (R_n(F), \circ)$. Then we have

i)$\langle a_{ij}, c_{lk} \rangle \mathcal{R}^* \langle b_{ij}, d_{lk} \rangle$ if and only if $a_{ij} \mathcal{R}^* b_{ij}$ and $c_{lk} \mathcal{R}^* d_{lk}$.

ii)$\langle a_{ij}, c_{lk} \rangle \mathcal{L}^* \langle b_{ij}, d_{lk} \rangle$ if and only if $a_{ij} \mathcal{L}^* b_{ij}$ and $c_{lk} \mathcal{L}^* d_{lk}$.

iii)$\langle a_{ij}, c_{lk} \rangle \mathcal{H}^* \langle b_{ij}, d_{lk} \rangle$ if and only if $a_{ij} \mathcal{H}^* b_{ij}$ and $c_{lk} \mathcal{H}^* d_{lk}$.

Proof. i) Suppose $\langle a_{ij}, c_{lk} \rangle \mathcal{R}^* \langle b_{ij}, d_{lk} \rangle$, then for $\langle x_{ij}, x_{lk} \rangle, \langle y_{ij}, y_{lk} \rangle \in S$ we have

$\langle x_{ij}, x_{lk} \rangle \langle a_{ij}, c_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle a_{ij}, c_{lk} \rangle \Leftrightarrow \langle x_{ij}, x_{lk} \rangle \langle b_{ij}, d_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle b_{ij}, d_{lk} \rangle$

$\Rightarrow \langle x_{ij} a_{ij}, x_{lk} c_{lk} \rangle = \langle y_{ij} a_{ij}, y_{lk} c_{lk} \rangle \Leftrightarrow \langle x_{ij} b_{ij}, x_{lk} d_{lk} \rangle = \langle y_{ij} b_{ij}, y_{lk} d_{lk} \rangle$

Consequently, we have $x_{ij} a_{ij} = y_{ij} a_{ij}$, $x_{lk} c_{lk} = y_{lk} c_{lk} \Leftrightarrow x_{ij} b_{ij} = y_{ij} b_{ij}$, $x_{lk} d_{lk} = y_{lk} d_{lk}$.

This implies that $a_{ij} \mathcal{R}^* b_{ij}$ and $c_{lk} \mathcal{R}^* d_{lk}$

Conversely, let $a_{ij} \mathcal{R}^* b_{ij}$ and $c_{lk} \mathcal{R}^* d_{lk}$, then there exists arbitrary elements $x_{ij}, y_{ij} \in \mathcal{M}_t(F)$ and $x_{lk}, y_{lk} \in \mathcal{M}_{t-1}(F)$ such that $x_{ij} a_{ij} = y_{ij} a_{ij} \Leftrightarrow x_{ij} b_{ij} = y_{ij} b_{ij}$ and $x_{lk} c_{lk} = y_{lk} c_{lk} \Leftrightarrow x_{lk} d_{lk} = y_{lk} d_{lk}$

It follows that

$\langle x_{ij}, x_{lk} \rangle \langle a_{ij}, c_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle a_{ij}, c_{lk} \rangle \Leftrightarrow \langle x_{ij}, x_{lk} \rangle \langle b_{ij}, d_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle b_{ij}, d_{lk} \rangle$

Thus ,$\langle a_{ij}, c_{lk} \rangle \mathcal{R}^* \langle b_{ij}, d_{lk} \rangle$.

Lemma :

Let $\langle a_{ij}, c_{lk}\rangle \epsilon\, S = (R_n(\text{F}), \circ)$. Then $\langle a_{ij}, c_{lk}\rangle \epsilon$ E(S) if and only if $a_{ij} \epsilon$ E($\mathcal{M}_t$(F)) and $c_{lk} \epsilon\, E(\mathcal{M}_{t-1}$(F))

Proof. Let $\langle a_{ij}, c_{lk}\rangle \epsilon$ E(S), then we have that

$\langle a_{ij}, c_{lk}\rangle \circ \langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle$

$\Rightarrow \langle a_{ij}c_{lk}, c_{lk}c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle$

Consequently,

$(a_{ij})^2 = a_{ij}$ (where $a_{ij} \epsilon\, \mathcal{M}_t$(F) is a square matrix)

$(c_{lk})^2 = c_{lk}$(where $c_{lk} \epsilon\, \mathcal{M}_{t-1}$(F) is a square matrix).

Thus $a_{ij} \epsilon$ E($\mathcal{M}_t$(F)) and $c_{lk} \epsilon\, E(\mathcal{M}_{t-1}$(F))

The converse of the proof can be easily verified.

Example : The following rhotrices are idempotents in R_5(F);

$$\left\langle \begin{array}{ccccc} & & 2 & & \\ & -1 & 0 & -2 & \\ 1 & 0 & 3 & 0 & -4 \\ & -2 & 0 & 4 & \\ & & -3 & & \end{array} \right\rangle = \left\langle \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle,$$

$$\left\langle \begin{array}{ccccc} & & 2 & & \\ & -1 & 3 & -2 & \\ 1 & 1 & 3 & -6 & -4 \\ & -2 & -2 & 4 & \\ & & -3 & & \end{array} \right\rangle = \left\langle \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix} \right\rangle,$$

$$\left\langle \begin{array}{ccccc} & & 2 & & \\ & -1 & 4 & -2 & \\ 1 & 12 & 3 & -1 & -4 \\ & -2 & -3 & 4 & \\ & & - & & \end{array} \right\rangle = \left\langle \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \right\rangle.$$

It is obvious that $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \epsilon$ E($\mathcal{M}_t$(F)) while $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \epsilon\, E(\mathcal{M}_{t-1}$(F))

Chapter 4

APPLICATIONS OF RHOTRIX THEORY

4.1 APPLICATION OF RHOTRIX THEORY IN CRYPTOGRAPHY USING MODIFIED VIGENÈRE CIPHER

METHODOLOGY

Encryption

Consider a plain text that is to be encrypted. Let m be the number of letters in the plain text.

Step 1: Convert the plain text into a stream of numerals using the scheme given below.

A	B	C	D	E	F	G	H	I	J	K	L
1	2	3	4	5	6	7	8	9	10	11	12
M	N	O	P	Q	R	S	T	U	V	W	X
13	14	15	16	17	18	19	20	21	22	23	24
Y	Z	_	?	!	.	,	‘	"			
25	26	27	28	29	30	31	32	33			

Step 2: Now, organize these streams of numerals into a rhotrix according to the specified cases

Case 1: If $m = (n^2+1)/2$, where n = 3,5,7…

Directly place the numerals into a rhotrix of order n. We call this rhotrix as the **message rhotrix**.

Case 2: If $m \neq (n^2+1)/2$

Then, repeat the plain text numerals sequentially until m attains the value (+1)/2

Now, convert this extended plain text into a stream of numerals using the same scheme previously used, and organize these numbers into a rhotrix of order n, to obtain the message rhotrix.

Step 3: Choose a key of length l > m. Convert the key into a stream of numerals using the above scheme.The key is repeated sequentially until its length matches that of the plain text. Now organize the equivalent numerals into a rhotrix of order n, called the **key rhotrix**.

Step 4: Multiply the message rhotrix with the key rhotrix to obtain the encrypted rhotrix.

Step 5: Convert this rhotrix to a stream of numbers. This is the encrypted code to be send to the receiver.

Decryption

Step 1: Place the encrypted stream of numbers into a rhotrix.

Step 2: Multiply the encrypted rhotrix with the inverse of the key rhotrix, we get the decrypted rhotrix, same as the message rhotrix.

Step 3: Convert this rhotrix to a stream of numbers.

Step 4: Convert the stream of numbers back to text using the original scheme. Remove the repeated segments of the plain text to obtain the original plain text.

ILLUSTRATION

Encryption

Consider the plain text I AM FINE !

Here, m = 11. Repeat the letters of the plain text until it contains 13 letters. Now the plain text to be encrypted becomes

I AM FINE !

Convert this plain text, to numbers using the scheme.

I   A M   F I N E   !   I

9 27 1 13 27 6 9 14 5 27 29 27 9

Now organise these numbers into a rhotrix.

$$\left\langle \begin{array}{ccccc} & & 9 & & \\ & 6 & 13 & 27 & \\ 29 & 5 & 9 & 27 & 1 \\ & 9 & 27 & 14 & \\ & & 27 & & \end{array} \right\rangle$$

This rhotrix is the message rhotrix.

Let 'BE' be the key used to encrypt the message. Repeat this key until the length of the key becomes the same as the length of the plain text to be encrypted. Convert this key to numbers using the same scheme.

B E B E B E B E B E B E B

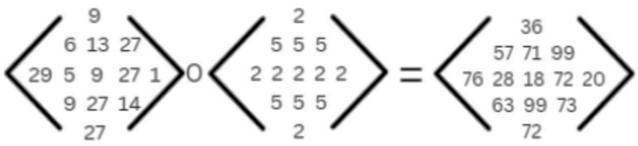
2 5 2 5 2 5 2 5 2 5 2 5 2

Now organise these numbers to a rhotrix.

$$\left\langle \begin{array}{ccccc} & & 2 & & \\ & 5 & 5 & 5 & \\ 2 & 2 & 2 & 2 & 2 \\ & 5 & 5 & 5 & \\ & & 2 & & \end{array} \right\rangle$$

This is the key rhotrix.

Now multiply the message rhotrix and key rhotrix.



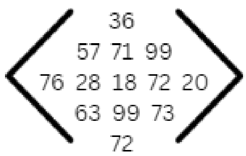
This is the encrypted rhotrix. Now convert this rhotrix into a stream of numbers.

36 99 20 71 72 57 18 73 28 99 76 63 72

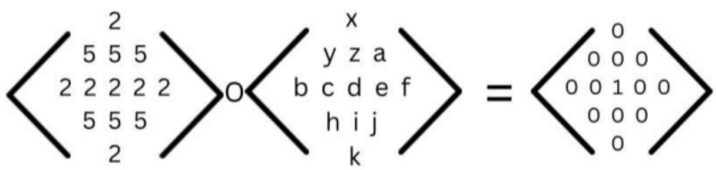
This is the encrypted code to be send to the receiver.

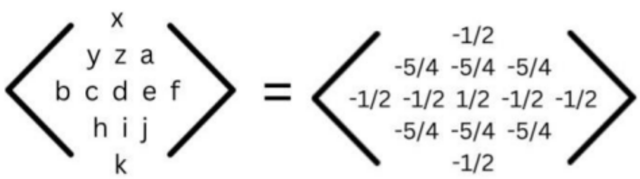
<u>Decryption</u>

Place the encrypted stream of numbers into a rhotrix.

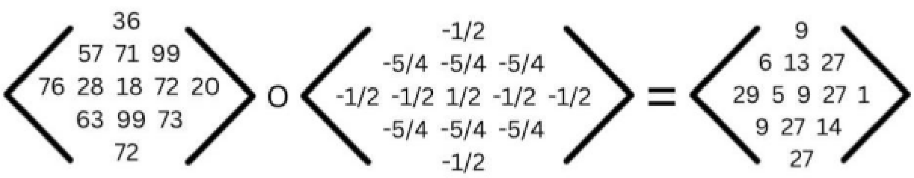


Find the inverse of the key rhotrix.





Now multiply the encrypted rhotrix with the inverse of the key rhotrix.



The decrypted rhotrix is obtained, same as the message rhotrix.
Now convert this rhotrix into a stream of numbers and convert it to text using the original scheme.

9 27 1 13 27 6 9 14 5 27 29 9 27

I A M F I N E ! I

Eliminate the repetitive stream of plain text. The original plain text is obtained.

I AM FINE !

The research demonstrates that the cryptosystem developed through rhotrix theory, utilizing a modified version of the Vigenère cipher, provides an effective framework for information encryption and decryption. The application of rhotrix theory within cryptographic systems has been shown to enhance security and reliability, offering a novel method for safeguarding data. The proposed cryptosystem, which employs rhotrix theory alongside a modified version of the Vigenère cipher using heart-oriented multiplication, proves to be an effective approach for enhancing the security of information encryption and decryption. This research area remains open for future exploration, particularly in investigating the use of row-column multiplication of rhotrices as an alternative to heart-oriented multiplication.

4.2 A NOVEL APPROACH TO CRYPTOGRAPHY THROUGH THE APPLICATION OF RHOTRIX THEORY

Encryption

The following method introduces a novel approach to cryptography, utilizing rhotrix theory to encrypt messages. This method is designed for messages in English and supports specific punctuation marks, including ‘_’, ‘.’, ‘,’, ‘?’, ‘!’ and ‘:’ in the message. Here, the symbol ‘_ ’ is used to represent the space character.

The following table shows the respective numbers representing the above-mentioned alphabets and symbols.

A	**0**		**Q**	**16**
B	**1**		**R**	**17**

C	**2**		**S**	**18**
D	**3**		**T**	**19**
E	**4**		**U**	**20**
F	**5**		**V**	**21**
G	**6**		**W**	**22**
H	**7**		**X**	**23**
I	**8**		**Y**	**24**
J	**9**		**Z**	**25**
K	**10**		**_**	**26**
L	**11**		**.**	**27**
M	**12**		**,**	**28**
N	**13**		**?**	**29**
O	**14**		**!**	**30**
P	**15**		**:**	**31**

Now, consider the message "HOW ARE YOU?". To encrypt this message using the proposed method, rewrite it as:

H O W _ A R E _ Y O U ?

Let k represents the total number of characters in the message to be encrypted, including the specified punctuation marks.

Here, k = 12.

The number of elements in an odd dimensional rhotrix of order n is $(n^2 + 1)/2$.

Hence to represent this message as an odd dimensional rhotrix, consider the inequality

$$\frac{n^2+1}{2} \geq k$$

Find the smallest odd number n that satisfies the above inequality. This value of n will then be used as the dimension for constructing the rhotrix. In this case,

$$\frac{n^2+1}{2} \geq 12$$

On solving we get n = 5 which is the rhotrix's dimension, $\frac{5^2+1}{2}=13 \; for \; n=5$

Now, let $q=\left(\frac{n^2+1}{2}\right)-k$

If q = 0, this indicates that there are no empty entries in the rhotrix formed, whereas, if q ≠ 0, it indicates that there are ' q ' empty spaces in the rhotrix.

These q spaces should be filled by the numerical equivalents of the letters in the following sequence:

" :XYZXYZXYZXYZ……"

where the first empty space is filled by ':', the second by 'X', the third by 'Y', and so forth. Note that the symbol ':' is reserved exclusively for this sequence and should not be used in the message to be encrypted.

In this case, q = 13−12 = 1, indicating that there is 1 empty entry. The message then becomes:

H	O	W	_	A	R	E	_	Y	O	U	?	:
7	14	22	26	0	17	4	26	24	14	20	29	31

To convert these values into a rhotrix R , begin by filling the horizontal axis (the mth row, which is the [(n +1)/2]th row). The remaining numbers should then be filled in the following order: (m - 1)th, (m + 1)th, (m - 2)th, (m + 2)th continuing alternately until the 1st and nth rows of the rhotrix R are filled.

In this case,

$$R=\left\langle \begin{array}{ccccc} & & 29 & & \\ & 17 & 4 & 26 & \\ 7 & 14 & 22 & 26 & 0 \\ & 24 & 14 & 20 & \\ & & 31 & & \end{array} \right\rangle$$

Here , heart of the rhotrix h(R) = 22

For encrypting the data, consider the equation

$$\lambda' = (\lambda + h(R))(mod\, 26)$$

where λ is each element in R . Thus by the above equation, we could form the modified Rhotrix R ′ by replacing each λ by λ′ where λ′ is each element in R ′. The following conditions also must be followed in order to form the new Rhotrix.

h(R) = h(R ′)

λ = λ′ if λ > 25.

In this case, $R' = \left\langle \begin{array}{ccccc} & & 29 & & \\ & 13 & 0 & 26 & \\ 3 & 10 & 22 & 26 & 22 \\ & 20 & 10 & 16 & \\ & & 31 & & \end{array} \right\rangle$

Next, rearrange these numbers column-wise into a single line and replace each number with its corresponding original value from Table 1. This process yields the final encrypted message.

3	**13**	**10**	**20**	**29**	**0**	**22**	**10**	**31**	**26**	**26**	**16**	**22**
D	**N**	**K**	**U**	**?**	**A**	**W**	**K**	**:**	**_**	**_**	**Q**	**W**

Therefore, in this case,the encrypted message is

DNKU?AWK:__QW

Decryption

Let us consider the encrypted message **“ DNKU?AWK:__QW ”.** Referring to Table 1, we obtain:

D	**N**	**K**	**U**	**?**	**A**	**W**	**K**	**:**	**_**	**_**	**Q**	**W**
3	**13**	**10**	**20**	**29**	**0**	**22**	**10**	**31**	**26**	**26**	**16**	**22**

To convert this data into an odd-dimensional rhotrix, the first step is to determine the dimension n using the same procedure applied in the encryption method.

Now, construct the rhotrix R ′ by entering the numbers column wise.

In this case k = 13 . So, n = 5 i.e., the rhotrix to be formed is of dimension 5.

Thus we obtain,

$$R' = \left\langle \begin{array}{ccccc} & & 29 & & \\ & 13 & 0 & 26 & \\ 3 & 10 & 22 & 26 & 22 \\ & 20 & 10 & 16 & \\ & & 31 & & \end{array} \right\rangle$$

Here h(R ′) = 22.

Now consider the equation, $\lambda = (\lambda' - h(R'))(mod\, 26)$

where λ′ is each element in R ′ .

Thus , Rhotrix R is formed by replacing each λ by λ′ in R ′ and following the conditions,

1. **h(R ′) = h(R)**
2. **λ = λ′ if λ′ > 25.**

Here in this case

$$R=\left\langle\begin{array}{ccccc} & & 29 & & \\ & 17 & 4 & 26 & \\ 7 & 14 & 22 & 26 & 0 \\ & 24 & 14 & 20 & \\ & & 31 & & \end{array}\right\rangle$$

To retrieve the data, take the horizontal axis of the rhotrix R as m ,ie, the {(n+1)/2}th row.

Now, enlist the data from the mth row followed by

(m - 1)th, (m + 1)th, (m - 2)th, (m + 2)th,....., 1st, nth rows respectively in a single line as follows :-

7	**14**	**22**	**26**	**0**	**17**	**4**	**26**	**24**	**14**	**20**	**29**	**31**

Now this numerical data is converted to alphabets and symbols from table 1 to obtain the decrypted message. Here,

7	**14**	**22**	**26**	**0**	**17**	**4**	**26**	**24**	**14**	**20**	**29**	**31**
H	**O**	**W**	**_**	**A**	**R**	**E**	**_**	**Y**	**O**	**U**	**?**	**:**

If the decrypted message ends with the sequence " **:XYZXYZXYZXYZ….** " , remove this sequence and replace the **'_'** with a space character to obtain the original message.

Thus, by referring to the table, the original message is successfully recovered as:

HOW ARE YOU?

This method successfully transforms an input message into an encrypted format by employing rhotrix theory. This process involves identifying the appropriate dimensions for the rhotrix, filling it with corresponding numerical equivalents, and finally rearranging the data to produce a secure, encoded message. The resulting encrypted message maintains the original structure while ensuring confidentiality.

This paper presents a novel approach to cryptography through the application of Rhotrix theory. The encryption and decryption techniques discussed here represent significant advancements in ensuring secure data transmission and protection. By incorporating a new method for encrypting special characters, including the space character, this approach guarantees that messages are securely formatted.

This methodology provides a scope for future research into the study of rhotrices, inviting further exploration and development in the field.

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