TM254317W

Reg. No :...............................

Name :.................................

MASTER'S DEGREE (C.S.S) EXAMINATION, MARCH 2025

2020, 2021, 2022 ADMISSIONS SUPPLEMENTARY

SEMESTER IV - CORE COURSE MATHEMATICS

MT4C16TM20 - Spectral Theory

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Let $T \in B(X,Y)$ and (x_n) be a sequence in X where X and Y are normed spaces. If $x_n \xrightarrow{w} x_0$. Prove that $Tx_n \xrightarrow{w} Tx_0$
2. If T is a contraction , show that T^n is a contraction.
3. Give an example of a bounded linear operator, which is not closed.
4. Define spectral radius. For any operator $T \in B(X,X)$ on a complex Banach space X , show that
   i) $r_\sigma(\alpha T) = |\alpha| r_\sigma(T)$
   ii) $r_\sigma(T^k) = [r_\sigma(T)]^k$
5. Show that an eigen space of T is invariant under T.
6. If $\|x - e\| < 1$. Show that x is invertible and $x^{-1} = e + \sum_1^\infty (e - x)^j$.
7. Define compact linear operator. Prove that if dim X = ∞ then the identity operator I: X→Y is not compact
8. Let T be a compact linear operator on a normed space X then for all $\lambda \neq 0$, nullspace of $T\lambda$ is finite dimensional
9. Define projection operator. For any projection on a Hilbert space H, prove that $\langle Px, x \rangle = \| Px \|^2$
10. Let S and T be bounded self adjoint linear operator on a complex Hilbert space H. If $S \geq 0$, show that $TST \geq 0$.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Prove that strong convergence implies weak convergence. Is the converse true? Justify your answer.
12. a) Define a closed linear operator. b) State and prove closed graph theorem.
13. Prove that the spectrum of a bounded linear operator T on a complex Banach space X is closed.
14. Let A be a Banach algebra with identity e , if $\|x\| < 1$.Prove that e-x is invertible and $(e - x)^{-1} = e + \sum_1^\infty x^j$
15. Show that $T: l^\infty \to l^\infty$ defined by $Tx = (\frac{\zeta_j}{j})$ is compact whenever $x = (\zeta_j) \in l^\infty$
16. Prove that totally bounded set are bounded. Illustrate by an example that the converse of this statement is not true.
17. Let *T:H→H* be a bounded linear self adjoint operator on a complex Hilbert space *H and let* $m = \inf_{\|x\|=1} \langle Tx, x \rangle$ and $M = \sup_{\|x\|=1} \langle Tx, x \rangle$ and $k = \sup_{\|x\|=1} |\langle Tx, x \rangle|$ .Prove that $\|T\| = k$.
18. Let T be a bounded self adjoint linear operator on a complex Hilbert space H. Show that T^*T is self adjoint and positive. Also show that spectrum of T^*T is real.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. a) Define contraction on a metric space. b) Prove that contraction T on a metric space X is continuous. c) Prove that T has precisely one fixed point where T is a contraction on a complete metric space X.
20. a) The resolvant set of a bounded linear operator T on a complex Banach space X is locally holomorphic on $\rho(T)$.
    b) If X is a nonempty Banach space and if $T \in B(X,X)$ then prove that $\sigma(T) \neq \phi$.

21. Let B ba a subset of a metric space X. Prove the following:

a) If B is relatively compact, then B is totally bounded.

b) If B is totally bounded and X is complete, then B is relatively compact.

c) If B is totally bounded, for every $\varepsilon>0$, there exists a finite ε-net $M_\varepsilon \subset B$.

d) If B is totally bounded, then B is separable.

22. a) Let $T: H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Then prove that $\lambda \in \rho(T)$ if and only if there exists $c > 0$ such that $\|T_\lambda x\| \geq c\|x\|$ for every $x \in H$.

b) Prove that all spectral values a bounded self adjoint linear operator on a complex Hilbert space are real.