TM254926E

Reg. No :...............................

Name :...................................

MASTER'S DEGREE (C.S.S) EXAMINATION, MARCH 2025

2020, 2021, 2022 ADMISSIONS SUPPLEMENTARY

MATHEMATICS SEMESTER IV - ELECTIVE COURSE

MT4E03TM20 - Analytic Number Theory

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Define Liouville's function. Show that it is completely multiplicative.
2. Prove that d(n) is odd if and only if n is a square.
3. Define Euler phi function. Find all integers n such that $\varphi(n)= n/3$.
4. Define Chebyshev's ψ-function and ϑ-function.
5. Calculate the highest power of 10 that divides 1000!
6. State and prove Fermat Little Theorem
7. Define a complete residue system modulo m. Give an example.
8. Find the remainder when 42^{73} is divided by 5.
9. Let g be a primitive root mod p, where p is an odd prime. Show that even powers of g are quadratic residues mod p.
10. Define (i) partition (ii) partition function and (iii) parts

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Prove that if f and g are multiplicative so is their Dirichlet product.
12. Prove that f = g if and only if $f_p(x) = g_p(x)$ for all primes p, where f and g are multiplicative.
13. Prove that there is a constant A such that $\sum_{p\leq x} \frac{1}{p} = \log\log x + A + O(\frac{1}{\log x}) \forall x \geq 2$.
14. If $0 < a < b$, prove that there exist an x_0 such that $\pi(ax) < \pi(bx)$ if $x \geq x_0$.
15. Prove : Given a prime p, let $f(x) = c_0 + c_1 x + \cdots c_n x^n$ be a polynomial of degree n with integer coefficients such that $c_n \not\equiv 0 \pmod p$.

    Then the polynomial congruence $f(x) \equiv 0 \pmod p$ has at most n solutions.
16. Find all positive integers n for which $n^{13} \equiv n \pmod{1365}$
17. Let g be a primitive root of an odd prime p. Prove that −g is also a primitive root of p if $p \equiv 1 \bmod 4$, but that exponent of (-g) modulo p = (p − 1)/2 if $p \equiv 3 \bmod 4$.
18. Prove or disprove : There exist primitive roots for prime moduli

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. a) Prove that two lattice points (a,b) and (m,n) are mutually visible if and only if a-m and b-n are relatively prime.

    b) Show that the set of lattice points visible from the origin has density $6/\pi^2$.

20. State the prime number theorem. If p_n denote the n^{th} prime, prove that the prime number theorem is equivalent to the asymptotic relation $\lim_{n \to \infty} \frac{p_n}{n \log n} = 1$

21. For any prime $p \geq 5$, show that $\sum_{k=1}^{p-1} \frac{(p-1)!}{k} \equiv 0 \pmod{p^2}$

22. State and prove the Euler's identity.