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Reg. No :...............................

Name :.................................

MASTER'S DEGREE (C.S.S) EXAMINATION, NOVEMBER 2024

2023 ADMISSIONS REGULAR

SEMESTER III - CORE COURSE MATHS

MT3C12TM20 - Functional Analysis

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Show that T: $R^2 \to R^2$ defined by T(x, y) = (ax, ay), where a is a fixed real number, is linear. Also find the range and null space of it.
2. Prove that every convergent sequence in a normed space is bounded.
3. If X is a compact metric space and $M \subset X$ is closed then prove that M is compact.
4. Let f be a linear functional on n- dimensional vector space X. Find the dimension of N(f).
5. If in an inner product space <x, u> = <x, v> , for all x. Prove that u = v.
6. Define orthogonal complement of a Hilbert space H. Prove that $Y^{\perp}$ is a closed subspace of H.
7. Show that the operator T : $H \to H'$ which is defined by T(z) = f(z) = < ., z > is a conjugate linear isometric bijection.
8. Let T : $H \to H$ be a bounded linear operator on a complex Hilbert space H and < Tx, x> is real for all x in H then prove that T is self adjoint.
9. Let X be a normed space and $x_0 \in X$ be such that $f(x_0) = 0$, for all f in X'. Prove that $x_0 = 0$.
10. Let Y be a closed subspace of a normed space X such that every f ∈ X' which is zero everywhere on Y is zero everywhere on whole space X. Show that Y = X.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Prove that a subspace Y of a Banach space X is complete if and only if Y is closed in X.
12. Give examples of a bounded and unbounded linear operators.
13. Define algebraically reflexive space. Prove that every finite dimensional vector space is algebraically reflexive.
14. Show that in an inner product space $x \perp y$ if and only if for any scalar α ,‖ x + αy ‖ = ‖ x - αy ‖.
15. a) State and Prove Bessel's Inequality. b) Give an example where we have strict equality in Bessel's Inequality.
16. Let H be a Hilbert space and H contains an orthonormal sequence that is total in H. Prove that H is separable.
17. Let X be a normed space and x_0 be any non zero element of X. Prove that there exists a bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = \dfrac{1}{\|x_0\|}$ and $\tilde{f}(x_0) = 1$.
18. Define adjoint operator. Prove that the adjoint operator $T^{\times}$ is linear, bounded and $\|T^{\times}\| = \|T\|$.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. a) State and prove F. Riesz's Lemma.

b) Let X be a normed space, the closed unit ball $M = \{x \in X \mid \|x\| \leq 1\}$ is compact. Show that X is finite dimensional .

20. a) Define B(X,Y) for normed spaces X and Y . b) Prove that B(X,Y) is a normed space. c) In addition if Y is a Banach space, prove that B(X,Y) is a Banach space.

21. a) Explain Gram-Schmidt process of orthonormalisation.

b) Orthonormalise 1st three terms of the sequence$\{x_0, x_1, x_2 \ldots\}$ where $x_t(j) = t^j$ on the interval [-1, 1] with respect to the inner product given by $<x, y> = \int_{-1}^{1} x(t)y(t)\mathrm{d}t.$

22. State and prove Hahn Banach theorem for complex vector spaces.