TM243933M

Reg. No :................................

Name :.................................

MASTER'S DEGREE (C.S.S) EXAMINATION, NOVEMBER 2024

2023 ADMISSIONS REGULAR

SEMESTER III - CORE COURSE MATHEMATICS

MT3C14TM20 - Differential Geometry

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Sketch the level set for n = 0,1,2 for the function $f(x_1,x_2,\dots,x_{n+1}) = x_{n+1}, c = -1,0,1,2.$
2. Prove that the extreme points of g on S are the Eigen vectors, where $g(x_1,x_2) = ax_1{}^2 + 2bx_1x_2 + cx_2{}^2$ and $S = f^{-1}(c), f(x_1,x_2) = x_1{}^2 + x_2{}^2$.
3. Let $\bar{X}$ and $\bar{Y}$ be smooth vector fields tangent to S along $\propto$, $\propto: I \to S$ and f be a smooth function along $\propto$, then prove that $(\bar{X}+\bar{Y})' = \bar{X}' + \bar{Y}'$.
4. Let $\bar{X}$ and $\bar{Y}$ be smooth vector fields tangent to S along $\propto$, $\propto: I \to S$ and f be a smooth function along $\propto$, then prove that $(\bar{X}.\bar{Y})' = \bar{X}.\bar{Y}' + \bar{Y}.\bar{X}'$.
5. Let $\bar{X}$ and $\bar{Y}$ be smooth tangent vector fields on S then prove that $D_{\bar{v}}(\bar{X}+\bar{Y}) = D_{\bar{v}}\bar{X} + D_{\bar{v}}\bar{Y}$
6. Define (i) One-form ω on U. (ii) The differential of f. (iii) The sum of 2 one norms.
7. Prove that, Let S be an n-surface in R^{n+1} oriented by the unit normal vector field $\bar{N}$, let $p \in S$ and $\bar{v} \in S_p$ then for any parameterized curve $\propto: I \to S$ with $\dot{\propto}(t_0) = \bar{v}$ for some $t_0 \in I, \dot{\propto}(t_0).\bar{N}(p) = L_p(\bar{v}).\bar{v}$
8. Define

   i. The image of $d\Psi_p$

   ii. The tangent space of Ψ at p.
9. Find the normal curvature κ(v) for each tangent direction v, the principal curvatures and principal curvature directions, and the Gauss- Kronecker and mean curvatures at the given point p of the given n-surface $f(x_1,x_2,\dots,x_{n+1}) = c$ oriented by $\frac{f}{||\nabla f||}$ where $x_1{}^2 + \left(\sqrt{x_2{}^2 + x_3{}^2} - 2\right)^2 = 1, p = (0,3,0).$
10. Find the normal curvature κ(v) for each tangent direction v, the principal curvatures and principal curvature directions, and the Gauss- Kronecker and mean curvatures at the given point p of the given n-surface $f(x_1,x_2,\dots,x_{n+1}) = c$ oriented by $\frac{\nabla f}{|\nabla f|}$ where $x_1{}^2 + \left(\sqrt{x_2{}^2 + x_3{}^2} - 2\right)^2 = 1, p = (0,1,0)$

Part B

II. Answer any Six questions. Each question carries 2 weight

(6x2=12)

11. Show that the two orientations on n-sphere $x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 + x_{n+1}{}^2 = r^2$ of radius *r>0* given by $\overline{N_1}(p) = (p, \frac{p}{r})$ and $\overline{N_2}(p) = (p, -\frac{p}{r})$.

12. Let $S \subseteq R^{n+1}$ be a connected n-surface in R^{n+1}, then prove that there exist on S exactly 2 smooth unit vector fields $\overline{N_1}$ and $\overline{N_2}$ and $\overline{N_2}(p) = -\overline{N_1}(p) \forall p \in S.$

13. Give an example of a parameterized curve having constant speed but it is not a geodesic.

14. Describe the spherical image when n = 1 and 2 of the given n-surface oriented by $\frac{\nabla f}{||\nabla f||}$, where f is a function defined by the LHS of the given equation, The sphere, $x_1{}^2 + x_2{}^2 + \cdots + x_{n+1}{}^2 = r^2, r > 0.$

15. Suppose $\bar{X}$ is a smooth unit vector field on an n-surface S in R^{n+1}. Show that $\nabla_{\bar{v}}\bar{X}$ is perpendicular to $\bar{X}(p) \forall v \in S_p, p \in S$. Show further that if $\bar{X}$ is a unit tangent vector field on S, then $D_{\bar{v}}\bar{X}$ is perpendicular to $\bar{X}(p)$.

16. Prove that the Weingarten map L_p is self-adjoint.

17. Find the normal curvature κ(v) for each tangent direction v, the principal curvatures and principal curvature directions, and the Gauss- Kronecker and mean curvatures at the given point p of the given n-surface $f(x_1, x_2, \ldots, x_{n+1}) = c$ oriented by $\frac{\nabla f}{|\nabla f|}$ where $(\frac{x_1}{a})^2 + (\frac{x_2}{a})^2 + (\frac{x_3}{a})^2 = 1, p = (a, 0,0)$

18. Prove that let S be an oriented n-surface in R^{n+1}, let $p \in S$, and let $\{k_1(p), \ldots k_n(p)\}$ be the principle curvatures of S at p with corresponding orthogonal principal curvature directions $\{v_1, v_2, \ldots, v_n\}$, then the normal curvature *k(v)* in the direction $v \in S_p(\|v\| = 1)$ is given by $k(v) = \sum_{i=1}^{n} k_i(p)\,(v.v_1)^2 = \sum_{i=1}^{n} k_i(p) cos^2\theta_i$, where $\theta_i = cos^{-1}(v.v_1)$.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. Prove that, Let $S = f^{-1}(C)$ be an n-surface in R^{n+1}, where $f:U \to R$ is such that $\nabla f(q) \neq 0 \; \forall \; q \in S$ and let $\bar{X}$ be smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of $\bar{X}$ such that $\alpha(t_0) \in S$ for some $t_0 \in I$ then $\alpha(t) \in S \; \forall t \in I$.

20. State and prove the Existence of a geodesic on the given surface through the given points with given velocity vector.

21. Prove that Let C be a connected oriented plane curve and let β:I→C be a unit speed, global parameterization of C then β is either one-one or periodic. Moreover β is periodic if and only if C is compact.

22.

    i. Prove that Let S be a compact connected oriented n-surface in R^{n+1}, then the Gauss-Kronecker curvature K(p) of S at p is non-zero for all $p \in S$ if and only if the second fundamental form ζp of S at p is definite for all $p \in S$.

    ii. Prove that on each component oriented n-surface S in R^{n+1} there exists a point p such that the second fundamental form at p is definite.