TM241587K

Reg. No :...............................

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MASTER'S DEGREE (C.S.S) EXAMINATION, NOVEMBER 2024
2020, 2021, 2022, 2023 ADMISSIONS SUPPLEMENTARY
SEMESTER I - CORE COURSE MATHEMATICS
MT1C01TM20 - Linear Algebra

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Define ordered basis and coordinate matrix relative to ordered basis.
2. Define subspace of a vector space. Show that the set of all Hermitian matrices is not a subspace of the space of all n x n matrices over C.
3. Let V be a finite dimensional vector space over the field F. Show that each basis for V* is the dual of some basis for V.
4. If S is any subset of a finite dimensional vector space V, then prove that $(S^{o})^{o}$ is the subspace spanned by S.
5. Define a linear transformation. If T is a linear transformation from a vector space V into W, show that T (0) = 0.
6. For a 2x2 matrix A over a field prove that det(I+A)=1+detA if and only if trace(A) = 0.
7. Check whether σ and τ are odd or even where σ and τ are permutations of degree 4 defined by $\sigma_1 = 2$, $\sigma_2 = 3$, $\sigma_3 = 4$, $\sigma_4 = 1$, $\tau_1 = 3$, $\tau_2 = 1$, $\tau_3 = 2$, $\tau_4 = 4$. Also find $\sigma\tau$ and $\tau\sigma$.
8. Let T be a linear operator on V and U be another linear operator on V such that TU = UT, then show that range of U and the null space of U are invariant under T.
9. Define a projection of a vector space V. Prove that any projection E is trivially diagonalizable
10. Prove that every matrix A such that $A^2 = A$ is similar to a diagonal matrix.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Suppose P is an n x n invertible matrix over F. Let V be an n dimensional vector space over F, and let $\mathfrak{B}$ be an ordered basis of V. Then show that there is a unique ordered basis $\mathfrak{B}^1$ of V such that $[\alpha]_{\mathfrak{B}} = P\,[\alpha]_{\mathfrak{B}'}$ and $[\alpha]_{\mathfrak{B}'} = P^{-1}\,[\alpha]_{\mathfrak{B}}$ for every vector $\alpha \in V$.
12. Let F be a subfield of complex numbers and let the matrix P = $\begin{bmatrix} -1 & 4 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 8 \end{bmatrix}$ be invertible. Let $\mathfrak{B}$ be the standard basis in F^3. Find a unique basis $\mathfrak{B}'$ such that $[\alpha]_{\mathfrak{B}} = P\,[\alpha]_{\mathfrak{B}'}$. Also find $[\alpha]_{\mathfrak{B}'}$ where $\alpha = (x_1, x_2, x_3)$. In particular express (3, 2, -8) in terms of $\mathfrak{B}'$.
13. Show that every n-dimensional vector space over the field F is isomorphic to the space F^n.
14. Let T be a linear transformation from V into W where V and W are finite dimensional vector spaces over the field F. Show that rank (T^t) = rank (T).
15. Let F be a field and let D be any alternating 3-linear function on 3 X 3 matrices over the polynomial ring F[x]. For A = $\begin{bmatrix} x & 0 & -x^2 \\ 0 & 1 & 0 \\ 1 & 0 & x^3 \end{bmatrix}$ show that D(A) = $(x^4 + x^2)$ D $(\epsilon_1, \epsilon_2, \epsilon_3)$ where $\epsilon_1, \epsilon_2, \epsilon_3$ denote the rows of the 3 X 3 identity matrix.

16. Show that the determinant function on 2 X 2 matrices A over K, a commutative ring with identity, is alternating and 2-linear as a function of the columns of A.

17. Prove. The minimal polynomial divides the characteristic polynomial for T where T is a linear operator on a finite dimensional vector space V.

18. Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is a characteristic vector of T.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. (a) Consider V the set of all pairs (x, y) of real numbers and F the field of real numbers. Define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c(x, y) = (cx, y)$. Is V a vector space?

(b) Show that the vectors α_1 = (1, 1, 0, 0), α_2 = (0, 0, 1, 1), α_3 = (1, 0, 0, 4), α_4 = (0, 0, 0, 2) form a basis for R^4. Find the coordinates of each of the standard basis vector in the ordered basis $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

20. (a) If A and B are n x n matrices over the field F, show that trace (AB) = trace (BA). Also show that similar matrices have the same trace.

(b) Let V and W be vector spaces over the field F with dimensions n and m respectively. For each pair of ordered bases $\mathfrak{B}, \mathfrak{B}'$ for V and W respectively, show that the function which assigns to a linear transformation T its matrix relative to $\mathfrak{B}, \mathfrak{B}'$ is an isomorphism between the space L (V, W) and the space of all m x n matrices over the field F.

21. (a) Prove that the determinant of the Vander-monde matrix $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is (b – a) (c – a) (c – b).

(b) Using Cramer's rule, solve the given system of linear equations over the field of rational numbers:

3x - 2y = 7

3y - 2z = 6

3z - 2x = -1

22. (a) Suppose T be a linear operator on the n dimensional vector space V and suppose that T has n distinct characteristic values. Prove that T is diagonalizable.

(b) Let V be a finite dimensional vector space over the filed F and let T be a linear operator on V. Show that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.