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Reg. No :................................

Name :..................................

MASTER'S DEGREE (C.S.S) EXAMINATION, NOVEMBER 2024

2024 ADMISSIONS REGULAR

SEMESTER I - CORE COURSE MATHEMATICS

MT1C01TM20 - Linear Algebra

Time : 3 Hours

Maximum Weight : 30

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Let U be the vector space of all 2 X 3 matrices over the field F. Form a basis for U. What is its dimension?
2. Consider V the set of pairs (x, y) of real numbers and F the field of real numbers. Is V with operations defined by $(x, y) + (x_1, y_1) = (x + x_1, 0)$ and $c(x, y) = (cx, 0)$ a vector space?
3. Define a linear functional. Give an example.
4. Suppose V is a finite dimensional vector space over a field F. Show that V and V** are isomorphic.
5. Consider the linear operator T on R^2 defined by $T(x_1, x_2) = (-x_2, x_1)$. Find the matrix of T in the standard ordered basis of R^2.
6. Let A be an $n \times n$ matrix. Show that det A = det A^t, where A^t is the transpose of A.
7. Check whether σ and τ are odd or even where σ and τ are permutations of degree 4 defined by $\sigma_1 = 2$, $\sigma_2 = 3$, $\sigma_3 = 4$, $\sigma_4 = 1$, $\tau_1 = 3$, $\tau_2 = 1$, $\tau_3 = 2$, $\tau_4 = 4$. Also find $\sigma\tau$ and $\tau\sigma$.
8. Define a projection of a vector space V. Prove that any projection E is trivially diagonalizable
9. If T is a linear operator on V, then show that range of T and the null space of T are also invariant under T.
10. Similar matrices have same minimal polynomial. Explain.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Show that row equivalent matrices have the same row space.
12. Find the coordinate matrix of the vector (1, 0 ,1) in the basis of C^3 consisting of vectors $(2i, 1, 0)$, $(2, -1, 1)$, $(0, 1+i, 1-i)$ in that order.
13. If V and W are finite dimensional vector space over a field F, prove that V and W are isomorphic if and only if dim V = dim W.
14. Consider the linear functionals on R^4 such that $f_1(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 2x_3 + x_4$, $f_2(x_1, x_2, x_3, x_4) = 2x_2 + x_4$ and $f_3(x_1, x_2, x_3, x_4) = -2x_1 - 4x_3 + 3x_4$. Find the subspace annihilated by these functionals.
15. Prove that an n × n matrix A over a commutative ring with identity, K, is invertible if and only if det A is invertible in K.
16. For a positive integer n and a field F prove that if σ is a permutation of degree n, the function $T(x_1, \dots, x_n) = (x_{\sigma_1}, \dots, x_{\sigma_n})$ is an invertible linear operator on F^n.
17. Prove. The minimal polynomial divides the characteristic polynomial for T where T is a linear operator on a finite dimensional vector space V.
18. Let V be a finite dimensional vector space and let W_1 be any subspace of V. Prove that there is a subspace W_2 of V such that $V = W_1 \oplus W_2$.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. (a) Consider V the set of all pairs (x, y) of real numbers and F the field of real numbers. Define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c(x, y) = (cx, y)$. Is V a vector space?

(b) Show that the vectors α_1 = (1, 1, 0, 0), α_2 = (0, 0, 1, 1), α_3 = (1, 0, 0, 4), α_4 = (0, 0, 0, 2) form a basis for R^4. Find the coordinates of each of the standard basis vector in the ordered basis ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$).

20. Let V and W be vector spaces such that dim V = n and dim W = m. Show that L (V, W) is a finite dimensional vector space with dimension mn.

21. (a) Prove that the determinant of the Vander-monde matrix $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is (b – a) (c – a) (c – b).

(b) Using Cramer's rule, solve the given system of linear equations over the field of rational numbers:

3x - 2y = 7

3y - 2z = 6

3z - 2x = -1

22. (a) Suppose V be a finite dimensional vector space over a field F and T be a linear operator on *V*. Then show that *T* is diagonalizable if and only if the minimal polynomial for T is a product of linear polynomials over F.

(b) Find an invertible real matrix *P* such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal where $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix}$.