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TM244564F

Reg. No :..............................

Name :..................................

MASTER'S DEGREE (C.S.S) EXAMINATION, MARCH 2024

2022 ADMISSIONS REGULAR

SEMESTER IV - CORE COURSE MATHEMATICS

MT4C16TM20 - Spectral Theory

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Show that T: $R^2 \rightarrow R$ defined by $(x_1,x_2) \rightarrow x_1$ is open.
2. Let $T \in B(X,Y)$ and (x_n) be a sequence in X where X and Y are normed spaces. If $x_n \xrightarrow{w} x_0$. Prove that $Tx_n \xrightarrow{w} Tx_0$
3. Give an example of a bounded linear operator, which is not closed.
4. Prove that point spectrum is the set of all eigen values of T.
5. Find the eigen values of the identity operator on a normed space X.
6. Let A be a complex Banach Algebra with identity e , prove that the set of all invertible elements of A is an open subset of A.
7. prove that $C(X,Y) \subset B(X,Y)$, where X and Y are normed spaces.
8. Let $T: l^2 \rightarrow l^2$ be defined by $x=(\xi_j)$, $y=(\eta_j)=Tx$, $\eta_{2k}=\xi_{2k}$ and $\eta_{2k-1}=0$ for $k=1, 2, 3,$ .... Find $N(T_\lambda^n)$.
9. Prove that two closed subspaces Y and V are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$
10. Show that if $T \geq 0$, then $(I+T)^{-1}$ exists.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Prove that strong convergence implies weak convergence. Is the converse true? Justify your answer.
12. a) Define a closed linear operator. b) State and prove closed graph theorem.
13. Prove that the spectrum of a bounded linear operator T on a complex Banach space X is closed.
14. Let A be a complex Banach algebra with identity e and $x \in A$, then prove that $\sigma(x)$ is compact.
15. Prove that a linear operator T: X→ Y is compact if and only if T maps every sequence (x_m) onto a sequence (Tx_m) which has a convergent subsequence where X and Y be normed spaces.
16. Prove that a compact linear operator from a normed space X into a Banach space Y has a compact extension on the completion of X.
17. Prove that the residual spectrum of a bounded self adjoint linear operator on a complex Hilbert space is empty.
18. If P_1 and P_2 are projection of H onto Y_1 and Y_2 respectiecely with $P_1P_2 = P_2P_1$. Prove that $P_1 + P_2 - P_1P_2$ of H onto $Y_1 + Y_2$.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. a) Define contraction on a metric space. b) Prove that contraction T on a metric space X is continuous. c) State and prove Banach Fixed Point Theorem.

20. a) Suppose X is a complex Banach space , $T \in B(X,X)$ and $p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots\ldots + \alpha_0$ ($\alpha_n \neq 0$) then prove that $p(\sigma(T)) = \sigma(p(T))$.

b) Show that eigen vectors $x_1, x_2, \ldots\ldots., x_n$ corresponding to different eigen values $\lambda_1, \lambda_2, \lambda_3, , \ldots\ldots\ldots\ldots, \lambda_n$ of a linear operator T on a vector space X constitute a linear independent set.

21. Let X be a metric space and $B \subset X$. Prove the following:

a) If B is relatively compact, then B is totally bounded.

b) If B is totally bounded and X is complete, then B is relatively compact.

c) If B is totally bounded, for every $\varepsilon > 0$, there exists a finite ε -net $M\varepsilon \subset B$.

d) If B is totally bounded, then B is separable.

22. a) Let $T: H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Prove that $\lambda \in \rho(T)$ if and only if there exists $c > 0$ such that $\|T_\lambda x\| \geq c\|x\|$ for every $x \in H$.

b) Prove that all spectral values a bounded self adjoint linear operator on a complex Hilbert space are real.