

TM244564F

Reg. No :

Name :

MASTER'S DEGREE (C.S.S) EXAMINATION, MARCH 2024

2022 ADMISSIONS REGULAR

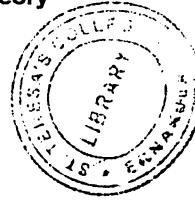
SEMESTER IV - CORE COURSE MATHEMATICS

MT4C16TM20 - Spectral Theory

Time : 3 Hours

Maximum Weight : 30

Part A



I. Answer any Eight questions. Each question carries 1 weight

(8x1=8)

1. Show that $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $(x_1, x_2) \rightarrow x_1$ is open.
2. Let $T \in B(X, Y)$ and (x_n) be a sequence in X where X and Y are normed spaces. If $x_n \xrightarrow{w} x_0$. Prove that $Tx_n \xrightarrow{w} Tx_0$.
3. Give an example of a bounded linear operator, which is not closed.
4. Prove that point spectrum is the set of all eigen values of T .
5. Find the eigen values of the identity operator on a normed space X .
6. Let A be a complex Banach Algebra with identity e , prove that the set of all invertible elements of A is an open subset of A .
7. prove that $C(X, Y) \subset B(X, Y)$, where X and Y are normed spaces.
8. Let $T: l^2 \rightarrow l^2$ be defined by $x = (\xi_j)$, $y = (\eta_j) = Tx$, $\eta_{2k} = \xi_{2k}$ and $\eta_{2k-1} = 0$ for $k=1, 2, 3, \dots$. Find $N(T_\lambda^n)$.
9. Prove that two closed subspaces Y and V are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$.
10. Show that if $T \geq 0$, then $(I+T)^{-1}$ exists.

Part B

II. Answer any Six questions. Each question carries 2 weight

(6x2=12)

11. Prove that strong convergence implies weak convergence. Is the converse true? Justify your answer.
12. a) Define a closed linear operator. b) State and prove closed graph theorem.
13. Prove that the spectrum of a bounded linear operator T on a complex Banach space X is closed.
14. Let A be a complex Banach algebra with identity e and $x \in A$, then prove that $\sigma(x)$ is compact.
15. Prove that a linear operator $T: X \rightarrow Y$ is compact if and only if T maps every sequence (x_m) onto a sequence (Tx_m) which has a convergent subsequence where X and Y be normed spaces.
16. Prove that a compact linear operator from a normed space X into a Banach space Y has a compact extension on the completion of X .
17. Prove that the residual spectrum of a bounded self adjoint linear operator on a complex Hilbert space is empty.
18. If P_1 and P_2 are projection of H onto Y_1 and Y_2 respectively with $P_1 P_2 = P_2 P_1$. Prove that $P_1 + P_2 - P_1 P_2$ of H onto $Y_1 + Y_2$.

Part C

III. Answer any Two questions. Each question carries 5 weight

(2x5=10)

19. a) Define contraction on a metric space. b) Prove that contraction T on a metric space X is continuous. c) State and prove Banach Fixed Point Theorem.

20. a) Suppose X is a complex Banach space, $T \in B(X, X)$ and $p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_0$ ($\alpha_n \neq 0$) then prove that $p(\sigma(T)) = \sigma(p(T))$.
- b) Show that eigen vectors x_1, x_2, \dots, x_n corresponding to different eigen values $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ of a linear operator T on a vector space X constitute a linear independent set.
21. Let X be a metric space and $B \subset X$. Prove the following:
- If B is relatively compact, then B is totally bounded.
 - If B is totally bounded and X is complete, then B is relatively compact.
 - If B is totally bounded, for every $\varepsilon > 0$, there exists a finite ε -net $M_\varepsilon \subset B$.
 - If B is totally bounded, then B is separable.
22. a) Let $T: H \rightarrow H$ be a bounded self adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Prove that $\lambda \in \rho(T)$ if and only if there exists $c > 0$ such that $\|T_\lambda x\| \geq c \|x\|$ for every $x \in H$.
- b) Prove that all spectral values a bounded self adjoint linear operator on a complex Hilbert space are real.

