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TM242572I

Reg. No :...............................

Name :...............................

MASTER'S DEGREE (C.S.S) EXAMINATION, MARCH 2024

2023 ADMISSIONS REGULAR

SEMESTER II - CORE COURSE

MT2C09TM20 - Measure Theory and Integration

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Let A be the set of irrational numbers in the interval [0,1]. Prove that $m^*(A) = 1$.
2. Explain the construction of the cantor set C.
3. Explain any two properties of outer measure.
4. Let f be an extended real valued function measurable on E and let f = g a.e. on E then, show that g is measurable on E.
5. Let g be a measurable real valued function defined on E and f a continuous real valued function defined on all of R. Show that the composition f o g is a measurable function on E.
6. Define a simple function. Derive the canonical representation of the simple function φ.
7. Define a measure space. Give an example.
8. Describe Dirac measure space.
9. Let (X, $\mathcal{M}$, μ) be a measure space and f be integrable over X. If A and B are disjoint measurable subsets of X, then show that $\int_{A \cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu$.
10. Let (X, $\mathcal{M}$, μ) be a measure space and f a measurable function on X. If f is bounded on X and vanishes outside a set of finite measure then show that f is integrable over X.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. Show that translate of a measurable set is measurable.
12. Prove that Lebesgue measure is countably additive.
13. Prove that integral of non-negative measurable function satisfies additivity over domain of integration.
14. For a non-negative measurable function on E, show that $\int_E f = 0$ if and only if f = 0 a.e. on E.
15. Show that union of a countable collection of measurable sets is measurable with respect to μ^*.
16. Let μ^* be an outer measure on 2^X. Then show that the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ - algebra.
17. Show that if $\psi \le \varphi$ a.e. on X, then $\int_X \psi\, d\mu \leqslant \int_X \varphi\, d\mu$ where φ and ψ are nonnegative simple functions on X and (X, $\mathcal{M}$, μ) is the measure space.
18. Let (X, $\mathcal{M}$, μ) be a measure space and $\{f_n\}$ a sequence of functions on X that is both uniformly integrable and tight over X. If $\{f_n\} \to f$ pointwise a.e. on X and the function f is integrable over X then show that $\lim_{n \to \infty} \int_E f_n\, d\mu = \int_E f\, d\mu$.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. (a) Prove that the outer measure of an interval is its length.

    (b) Show that given any set A and any $\varepsilon > 0$, there exists an open set O containing A such that $m^*(O) \leq m^*(A) + \varepsilon$.

20. (a)State and prove Fatou's Lemma.

    (b) Let f and g be integrable over E. Show that for any α and β, the function $\alpha f + \beta g$ is integrable over E and $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$. Moreover if f $\leq$ g on E, then show that $\int_E f \leq \int_E g$.

21. (a)State and prove Jordan Decomposition theorem.

    (b) For a Lebesgue measurable set E, define $\nu(E) = \int_E f\, dm$ where f is a real valued function that is Lebesgue integrable over $\mathbb{R}$. Show that ν is a signed measure on the measurable space $(\mathbb{R}, \mathcal{L})$. Also find a Hahn decomposition of $\mathbb{R}$ with respect to ν and Jordan decomposition of ν.

22. State and prove Fubini's Theorem.