1.7.4

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Name :.....

MASTER'S DEGREE (C.S.S) EXAMINATION, MARCH 2024 2023 ADMISSIONS REGULAR SEMESTER II - CORE COURSE MT2C08TM20 - Advanced Complex Analysis

Time: 3 Hours Maximum Weight: 30

Part A

I. Answer any Eight questions. Each question carries 1 weight

(8x1=8)

- 1. Prove that a non-constant Harmonic function has neither a maximum nor a minimum in its region of definition.
- 2. Show that log r is a harmonic function and that any harmonic function which depends only on r will be of the form a log r + b.
- 3. Derive the most general form of an entire function with a finite number of zeros.
- 4. Show that a necessary and sufficient condition for the absolute convergence of the product $\prod_{1}^{\infty} (1 + a_n)$ is the convergence of the series $\sum_{1}^{\infty} |a_n|$.
- 5. Prove: (a) $\Gamma(1) = 1$, (b) $\Gamma(n) = (n-1)!$
- 6. Briefly describe the zeros of Zeta function.
- 7. Prove that there are infinitely many primes.
- 8. State and prove a characterization of normal families.
- 9. Define the Weierstrass & -function.
- 10. Describe the fundamental period parallelogram in the context of elliptic functions.

Part B

II. Answer any Six questions. Each question carries 2 weight

(6x2=12)

- 11. If u(z) is harmonic in $|z| \le \rho$ and $U(\rho e^{i\theta}) \ge 0$ then show that $\frac{\rho r}{\rho + r}$ $U(0) \le U(z) \le \frac{\rho + r}{\rho r}$ U(0) where $|z| = r < \rho$.
- 12. Define a subharmonic function. State and prove a necessary and sufficient condition for a continuous function to be subharmonic.
- 13. Find the Taylor series expansion of $\frac{1}{z}$ at z = 1 and z = 2i.
- Derive the Laurent series expansions of $\frac{1}{(z-1)(z-2)}$ about z=1 and mention the regions in which those expansions are valid.
- 15. State and prove the theorem on the boundary behaviour of a topological mapping.
- 16. Show that the zeta function can be extended to a meromorphic function in the whole plane whose only pole is a simple pole at s=1 with residue 1.



- 17. Show that ζ (z + w₁) = ζ (z) + η_1 and ζ (z + w₂) = ζ (z) + η_2 where η_1 and η_2 are constants.
- 18. Show that any even elliptic function with periods w₁ and w₂ can be expressed in the form c

$$\prod_{k=1}^{n} \frac{\mathcal{P}(z) - \mathcal{P}(a_{k})}{\mathcal{P}(z) - \mathcal{P}(b_{k})}$$
 where c is a constant provided 0 is neither a zero nor a pole.

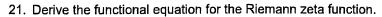
Part C

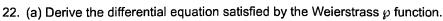
III. Answer any Two questions. Each question carries 5 weight

(2x5=10)

- 19. (a) State and Prove Schwarz theorem.
 - (b) Let $U_n(z)$ be a sequence of functions defined and harmonic in a region Ω_n and let Ω be a region such that every point in Ω has a neighbourhood contained in all but a finite number of the Ω_n . Assume that in this neighbourhood $U_n(z) \leq U_{n+1}(z)$ as soon as n is sufficiently large. Then show that there are only two possibilities: either $U_n(z)$ tends uniformly to $+\infty$ on every compact subset of Ω , or $U_n(z)$ tends to a harmonic limit function U(z) in Ω , uniformly on compact sets.
- 20. (a) State and prove Mittag Leffler theorem.

(b) Show that
$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$
.





(b) Show that
$$\mathscr{D}'(z) = -\frac{\sigma(2z)}{(\sigma(z))^4}$$
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