15 . 4 OC 15/4/2024

TM242851N

Reg. No :...............................

Name :.................................

MASTER'S DEGREE (C.S.S) EXAMINATION, MARCH 2024

2023 ADMISSIONS REGULAR

SEMESTER II - CORE COURSE

MT2C08TM20 - Advanced Complex Analysis

Time : 3 Hours **Maximum Weight : 30**

Part A

I. Answer any Eight questions. Each question carries 1 weight **(8x1=8)**

1. Prove that a non-constant Harmonic function has neither a maximum nor a minimum in its region of definition.
2. Show that log r is a harmonic function and that any harmonic function which depends only on r will be of the form a log r + b.
3. Derive the most general form of an entire function with a finite number of zeros.
4. Show that a necessary and sufficient condition for the absolute convergence of the product $\prod_{1}^{\infty}(1 + a_n)$ is the convergence of the series $\sum_{1}^{\infty} | a_n |$.
5. Prove: (a) $\Gamma(1) = 1$, (b) $\Gamma(n) = (n-1)!$
6. Briefly describe the zeros of Zeta function.
7. Prove that there are infinitely many primes.
8. State and prove a characterization of normal families.
9. Define the Weierstrass $\wp$-function.
10. Describe the fundamental period parallelogram in the context of elliptic functions.

Part B

II. Answer any Six questions. Each question carries 2 weight **(6x2=12)**

11. If u(z) is harmonic in $|z| \leq \rho$ and $U(\rho e^{i\theta}) \geq 0$ then show that $\frac{\rho - r}{\rho + r} U(0) \leq U(z) \leq \frac{\rho + r}{\rho - r} U(0)$ where $|z| = r < \rho$.
12. Define a subharmonic function. State and prove a necessary and sufficient condition for a continuous function to be subharmonic.
13. Find the Taylor series expansion of $\frac{1}{z}$ at z = 1 and z = 2i.
14. Derive the Laurent series expansions of $\frac{1}{(z-1)(z-2)}$ about z = 1 and mention the regions in which those expansions are valid.
15. State and prove the theorem on the boundary behaviour of a topological mapping.
16. Show that the zeta function can be extended to a meromorphic function in the whole plane whose only pole is a simple pole at s=1 with residue 1.

17. Show that $\zeta(z + w_1) = \zeta(z) + \eta_1$ and $\zeta(z + w_2) = \zeta(z) + \eta_2$ where η_1 and η_2 are constants.

18. Show that any even elliptic function with periods w_1 and w_2 can be expressed in the form $c \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$ where c is a constant provided 0 is neither a zero nor a pole.

Part C

III. Answer any Two questions. Each question carries 5 weight **(2x5=10)**

19. (a) State and Prove Schwarz theorem.

    (b) Let $U_n(z)$ be a sequence of functions defined and harmonic in a region Ω_n and let Ω be a region such that every point in Ω has a neighbourhood contained in all but a finite number of the Ω_n. Assume that in this neighbourhood $U_n(z) \leq U_{n+1}(z)$ as soon as n is sufficiently large. Then show that there are only two possibilities: either $U_n(z)$ tends uniformly to $+\infty$ on every compact subset of Ω, or $U_n(z)$ tends to a harmonic limit function $U(z)$ in Ω, uniformly on compact sets.

20. (a) State and prove Mittag - Leffler theorem.

    (b) Show that $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$.

21. Derive the functional equation for the Riemann zeta function.

22. (a) Derive the differential equation satisfied by the Weierstrass $\wp$ function.

    (b) Show that $\wp'(z) = -\frac{\sigma(2z)}{(\sigma(z))^4}$.