Project Report

On

CONVEX GEOMETRY

Submitted

in partial fulfilment of the requirements for the award of the degree

of

BACHELOR OF SCIENCE

in

MATHEMATICS

by

ROSE ANNA DENNIS

(Register No. AB21AMAT027)

Under the Supervision of

Ms. RITTY JACOB

DEPARTMENT OF MATHEMATICS AND STATISTICS

ST. TERESA'S COLLEGE (AUTONOMOUS)

ERNAKULAM, KOCHI – 682011

MARCH 2024

ST. TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM

CERTIFICATE

This is to certify that the dissertation entitled, **CONVEX GEOMETRY** is a bonafide record of the work done by **Ms. ROSE ANNA DENNIS** under my guidance as partial fulfillment of the award of the degree of **Bachelor of Science in Mathematics** at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of thiswork has been submitted for any other degree elsewhere.

.

Date:

Place: Ernakulam

Ms. RITTY JACOB
Assistant Professor,
Department of Mathematics,
St. Teresa's College (Autonomous),
Ernakulam.

Dr. URSALA PAUL

Assistant Professor and Head,
Department of Mathematics,
St. Teresa's College (Autonomous),
Ernakulam.

ST. TERESA'S COLLEGE (AUTONOMOUS) DEPARTMENT OF MATHEMATICS AND STATISTICS ERNAKULAM

External Examiners

1: .... 2: ....

DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Ms. RITTY JACOB, Assistant Professor, Department of Mathematics and Statistics, St. Teresa's College (Autonomous), Ernakulam and has not been included in any other project submitted previously for theaward of any degree.

Ernakulam

Date:25-03-2024

ROSE ANNA DENNIS

REGISTER NO. AB21AMAT027

ACKNOWLEDGEMENT

I take this opportunity to express my sincere gratitude towards Ms. RITTY JACOB of the Department of Mathematics and Statistics of St. Teresa's college who encouraged me to carry out this work. Her continuous invaluable knowledgeable guidance throughout this study helped me to complete the work up to this stage.

Ernakulam
Date: 25-03-2024

ROSE ANNA DENNIS
REGISTER NO. AB21AMAT027

CONTENTS

1
INTRODUCTION

Convex geometry is a branch of mathematics that delves into the intricate properties and relationships associated with convex sets, which hold significant importance across various mathematical disciplines and practical applications. A convex set is defined as a subset of a vector space in which the line segment connecting any two points within the set lies entirely within the set itself. This seemingly simple definition gives rise to a rich and fascinating field that encompasses a wide range of concepts and theorems.

One of the central concepts in convex geometry is the convex hull. The convex hull of a set is the smallest convex set that contains the original set. It can be thought of as the "envelope" that wraps around the set while maintaining the convexity property. Understanding convex hulls is crucial in various areas, including optimization, computational geometry, and operations research.

Polytopes, which are higher-dimensional generalizations of polygons and polyhedra, are another key element of convex geometry. Polytopes are convex sets with additional structure, often defined by a set of linear inequalities. They provide a geometric framework for studying combinatorial and algebraic properties and have connections to areas such as linear programming and discrete geometry.

Carathéodory 's theorem, a cornerstone in convex geometry, offers a succinct insight into the structure of convex sets. Originating from the work of Constantin Carathéodory, the theorem delineates that any point within the convex hull of a set in R^n can be expressed as a convex combination of at most $n+1$ points from the set.

This fundamental result significantly simplifies the characterization of points in convex hulls, reducing the complexity of geometric and optimization problems. With applications spanning computational geometry, optimization theory, and convex analysis, Carathéodory's theorem stands as a key pillar in understanding the geometric properties and practical implications of convex sets, contributing to diverse mathematical disciplines and problem-solving methodologies.

Beyond these fundamental concepts, convex geometry has far-reaching implications in optimization problems. Convex optimization involves the task of minimizing (or maximizing) a convex function over a convex set, making it a powerful tool in fields like machine learning, signal processing, and operations research. The beauty of convex optimization lies in the existence of efficient algorithms and the guarantee of finding global optima.

Functional analysis, a branch of mathematics dealing with vector spaces of functions, also draws extensively from convex geometry. Convex sets play a crucial role in the study of Banach spaces, where the convexity of the unit ball is a defining property. This connection between convex geometry and functional analysis provides a deep understanding of spaces of functions and their properties.

Convex geometry is a captivating field that explores the geometric properties and relationships inherent in convex sets. From convex hulls and polytopes to the profound implications in optimization and functional analysis, convex geometry weaves a rich tapestry of mathematical concepts with broad applications. Its impact extends far beyond theoretical realms, making it a cornerstone in various scientific and technological advancements.

2

CONVEX SETS

A set C is convex if the line segment between any two points in C lies in C, i.e. $\forall x_1, x_2 \in C, \forall \theta \in [0,1]$

$$\theta x_1 + (1 - \theta) x_2 \in C.$$

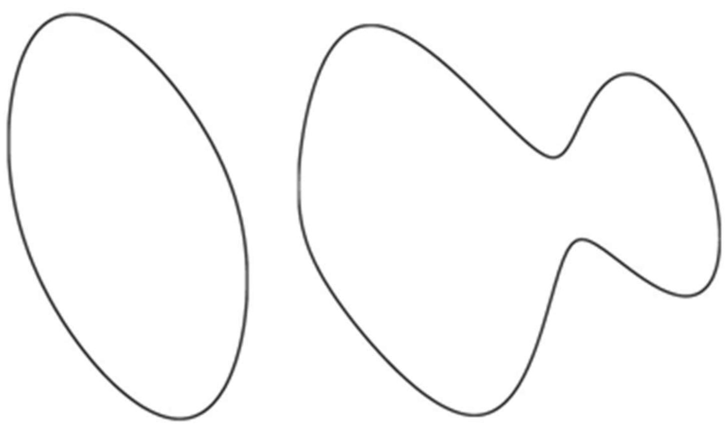
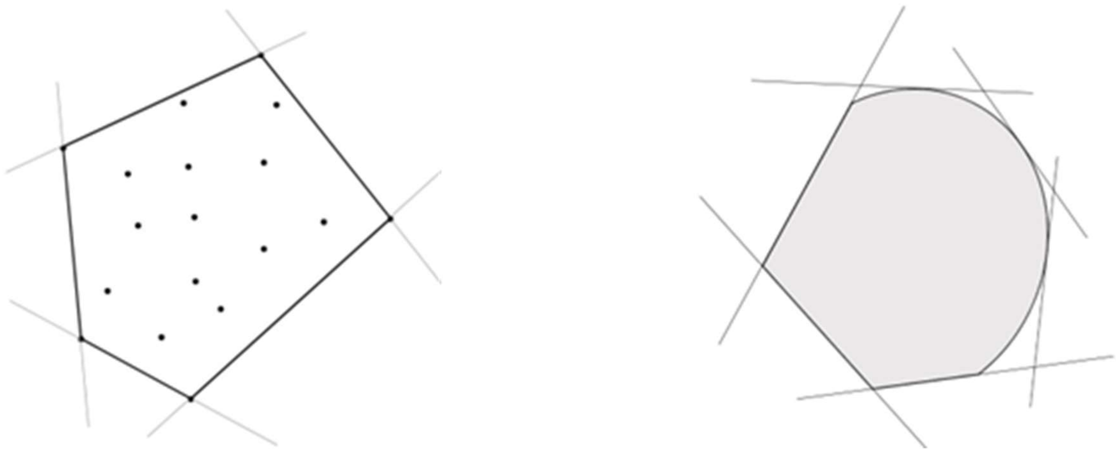


Figure 1: Example of a convex set (left) and a non-convex set (right).

Simple examples of convex sets are:

- The empty set $\emptyset$, the singleton set $\{x_0\}$, and the complete space R^n;
- Lines $\{a^T x = b\}$, line segments, hyperplanes $\{A^T x = b\}$, and halfspaces $\{A^T x \leq b\}$;
- Euclidian balls.

We can generalize the definition of a convex set above from two points to any number of points n. A convex combination of points $x_1,x_2,...,x_k \in C$ is any point of form $\theta_1 x_1 + \theta_2 x_2 + ... + \theta_k x_k$, where $\theta_i \geq 0, i = = 1$. Then, a set C is convex iff any convex combination of points in C is in C.



(a) (b)

Figure 2: (a) Representation of a convex set as the convex hull of a set of points.
(b) Representation of a convex set as the intersection of a (possibly infinite) number of halfspaces.

We can take this even further to infinite countable sums: C convex iff
$\forall x_i \in C, \theta_i \geq 0, i = 1, 2.., \sum_{i=1}^{\infty} \theta_i = 1$:

$$\sum_{i=1}^{\infty} \theta_i x_i \in C$$

if the series converges.

Most generally, C is convex iff for any random variable X over C, $P(X \in C) = 1$, its expectation is also in C:

$$E(X) \in C.$$

2.1 CONVEX SET

A convex set is a set in which, for any two points within the set, the straight line segment connecting them lies entirely within the set. In other words, a set is convex if, given any two points A and B in the set, the entire line segment AB is also in the set. This property ensures that the set contains all points along the shortest path between any two of its points. Convex sets have applications in various fields, including optimization, geometry, and economics.

A set C is considered convex if, for all Y and Z belonging to C, and for any value of μ between 0 and 1, the expression $\mu Y + (1 - \mu)Z$ also belongs to C. This condition defines a convex combination of Y and Z. In the context of real finite-dimensional Euclidean vector spaces or matrices, this combination represents the closed line segment connecting Y and Z. Therefore, a set is convex if the line segment between any two points in the set is also within the set, implying that convex sets are connected sets. It's worth noting that a convex set may or may not include the origin 0.

2.2 SUBSPACE

A nonempty subset R of real Euclidean vector space R^n is called a subspace if every vector of the form $\alpha x + \beta y$, for $\alpha,\beta \in R$, is in R whenever vectors x and y are.
A subspace is a convex set containing the origin.Any subspace is therefore open in the sense that it contains no boundary, but closed in the sense R+R=R

2.3 LINEAR INDEPENDENCE

Arbitrary given vectors in Euclidean space $\{\Gamma_i \in R^n, i=1 \dots N\}$ are linearly independent if and only if, for all $\zeta \in R^N$ ($\zeta i \in R$)

$$\Gamma_1 \zeta_1 + \cdot \cdot \cdot + \Gamma_{N-1} \zeta_{N-1} - \Gamma_N \zeta_N = \mathbf{0}$$

has only the trivial solution $\zeta = 0$; in other words, iff no vector from the given set can be expressed as a linear combination of those remaining.
Geometrically, two nontrivial vector subspaces are linearly independent iff they intersect only at the origin.

2.4 ORTHANT

Orthant is the name given to a closed convex set that is the higher-dimensional generalization of quadrant from the classical Cartesian partition of R^2; a Cartesian cone. The most common is the nonnegative orthant R^n_+ or $R^{n\times n}_+$ (analogue to quadrant I) to which membership denotes nonnegative vector- or matrix-entries respectively

$$R^n_+ \triangleq \{x \in R^N | x_i \geq 0\ \forall i\}$$

2.5 DIMENSION

Dimension of an arbitrary set $\mathcal{Z}$ is Euclidean dimension of its affine hull

$$\dim \mathcal{Z} \triangleq \dim \operatorname{aff} \mathcal{Z} = \dim \operatorname{aff}(\mathcal{Z} - s),\ s \in \mathcal{Z}$$

the same as dimension of the subspace parallel to that affine set aff Z when nonempty. Hence dimension (of a set) is synonymous with affine dimension.

2.6 AFFINE SET

A nonempty affine set is any subset of R^n that is a translation of some subspace. Any affine set is convex and open in the sense that it contains no boundary.

2.7 EMPTY SET AND EMPTY INTERIOR

Emptiness $\emptyset$ of a set is handled differently than interior in the classical literature. It is common for a nonempty convex set to have empty interior; e.g., paper in the real world: An ordinary flat sheet of paper is a nonempty convex set having empty interior in R^3 but non-empty interior relative to its affine hull.

2.8 THEOREMS

Theorem 1: Intersection of Convex Sets

Statement: The intersection of convex sets is convex. Convex $\bigcap^i C_i \equiv \bigcap^i$convex($C_i$)

Proof:

Let x, y be in $\bigcap^i C_i$. Since x, y are in every C_i, the convex combination (1-t)x + ty is also in every C_i, making it part of $\bigcap^i C_i$. Thus $\bigcap^i C_i$ is convex.

Theorem 2: Convexity of Vector Sum and Difference

Statement:

1. The vector sum (Minkowski sum) of convex sets C_1 and C_2 is convex.
   Convex($C_1 + C_2$) = convex(Minkowski_Sum($C_1 + C_2$))
2. The vector difference of convex sets C_1 and C_2 is convex.
   Convex($C_1 - C_2$) = convex (Vector Difference (C_1 , C_2))

Proof (for 1):

Let x = a+b and y = c + d where a, b ∈ C_1 and c, d ∈C_2.

The convex combination $(1-t)x+ty$ is$((1-t)a+tc)+((1-t)b+td)$, which belongs to C_1+ C_2.

Proof (for 2):

Let x, y be in C_1 - C_2. The convex combination $(1-t)(x-y)+ty$ is in C_1 - C_2, demonstrating the convexity of the vector difference.

Theorem 3: Convexity of Cartesian Product

Statement:

The Cartesian product of convex sets C_1 and C_2 is convex.

Convex ($C_1 \times C_2$) ≡ convex(Cartesian Product (C_1 , C_2))

Proof:

Let (x_1, y_1), (x_2, y_2) be in $C_1 \times C_2$. The convex combination $(1-t)x+ty$ is $((1-t)x_1+tx_2,(1-t)y_1+ty_2)$, which belongs to $C_1 \times C_2$.

Theorem 4: Convexity under Operations

Statement:

Convexity is preserved under scaling, rotation/reflection, and translation of convex sets.

Convex(Operation(T, C)) ≡ T(C)

Proof (for translation):

Let C be convex, and α be a translation vector. The convex combination $(1-t)x+ty$ in C becomes $(1-t)(x+\alpha)+t(y+\alpha)$ in $C+\alpha$, preserving convexity.

Theorem 5: Inverse Image Theorem for Convex Sets under Affine Functions

Statement:

Let f: $R^{p\times k} \rightarrow R^{m\times n}$ be an affine function, expressed as f(X) = AX + B, where A is an $m \times p$ matrix, X is a $p \times k$ matrix, and B is an $m \times n$ matrix. If C ⊆ $R^{p\times k}$ is a convex set, then the Image of C under f, denoted as f(c), is convex. Additionally, the inverse image of any convex set F⊆ $R^{m\times n}$ under f. denoted as f^{-1} (F), is convex.

Proof:

1. Image Convexity:

Let X_1, X_2∈C and $\lambda \in [0, 1]$.

Since C is convex, $\lambda X_1 + (1-\lambda)X_2 \in$ C. Consider the images $f(X_1)$ and $f(X_2)$ under f: $f(X_1)$=A(X_1)+B, $f(X_2)$= AX_2+ B

By the linearity of f, we have:

$f(\lambda X_1 + (1-\lambda)X_2) = A(\lambda X_1 + (1-\lambda)X_2) + B = \lambda f(X_1) + (1-\lambda)f(X_2)$

Since$(\lambda X_1 + (1-\lambda)X_2 \in C$ and $\lambda f(X_1) + (1-\lambda)f(X_2)$ is a convex combination of $f(X_1)$ and $f(X_2)$, it follows that $f(\lambda X_1 + (1-\lambda)X_2) \in f(C)$.

Therefore, $f(C)$ is convex.

2. Inverse Image Convexity:

Let F⊆ $R^{m\times n}$ be a convex set, and consider X_1, $X_2 \in f^{-1}$(F).

This implies that $f(X_1)$, $f(X_2)$ ∈(F).

Again, by the linearity of f, for any $\lambda \in [0, 1]$, we have:

$f(\lambda X_1 + (1-\lambda)X_2) = A(\lambda X_1 + (1-\lambda)X_2) + B = \lambda f(X_1) + (1-\lambda)f(X_2)$

Since $\lambda f(X_1) + (1-\lambda)f(X_2)$ is in F (as F is convex), it follows that $(\lambda X_1 + (1-\lambda)X_2) \in f^{-1}$ (F).

Therefore, $f^{-1}(F)$ is convex.

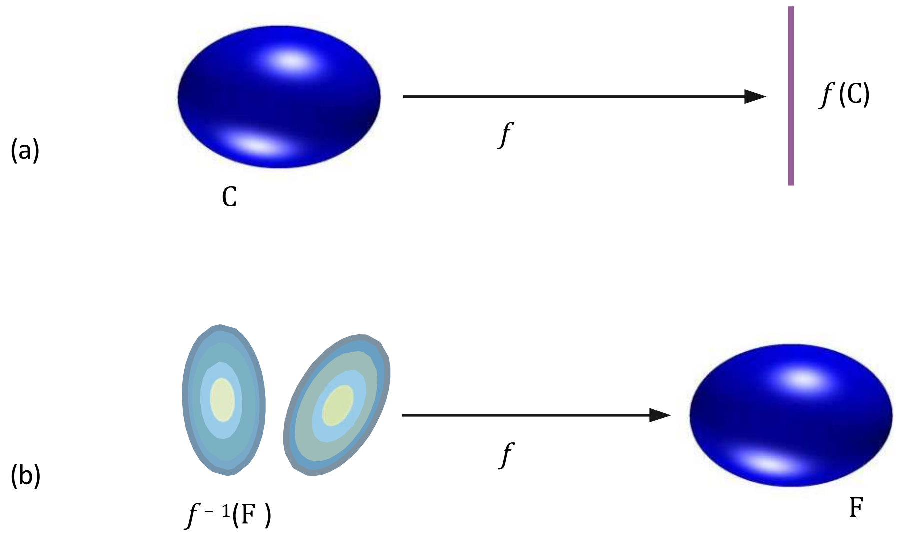


Figure 3: (a) Image of convex set in domain of any convex function f is convex, but there is no converse. (b) Inverse image under convex function f.

In particular, any affine transformation of an affine set remains affine. Inverse of any affine transformation, whose image is nonempty and affine, is affine. Ellipsoids are invariant to any affine transformation.

Although not precluded, this inverse image theorem does not require a uniquely invertible mapping f .

Each converse of this two-part theorem is generally false; i.e., given f affine, a convex image f(C) does not imply that set C is convex, and neither does a convex inverse image $f^{-1}(F)$ imply set F is convex. A counterexample, invalidating a converse, is easy to visualize when the affine function is an orthogonal projector.

3

POLYTOPES AND HULLS

3.1 CONVEX POLYTOPES

3.1.1 CONVEXITY IN POLYTOPES

A convex polytope is, by definition, a convex set. This means that for any two points inside the polytope, the line segment connecting them is also entirely contained within the polytope.

3.1.2 REGULAR POLYTOPES

A regular polytope is a polytope whose symmetry group acts transitively on its flags,

thus giving it the highest degree of symmetry. In particular, all its elements or j-faces (for all $0 \leq j \leq n$, where n is the dimension of the polytope) — cells, faces and so on — are also transitive on the symmetries of the polytope, and are themselves regular polytopes of dimension $j \leq n$.

Three classes of regular polytopes exist in every dimension:

- Regular simplex
- Measure polytope (Hypercube)
- Cross polytope (Orthoplex)

Any other regular polytope is said to be exceptional.

In one dimension, the line segment simultaneously serves as the 1-simplex, the 1-hypercube and the 1-orthoplex.

In two dimensions, there are infinitely many regular polygons, namely the regular n-sided polygon for $n \geq 3$. The triangle is the 2-simplex. The square is both the 2-hypercube and the 2-orthoplex. The n-sided polygons for $n \geq 5$ are exceptional.

In three and four dimensions, there are several more exceptional regular polyhedra and 4-polytopes.

In five dimensions and above, the simplex, hypercube and orthoplex are the only regular polytopes. There are no exceptional regular polytopes in these dimensions.

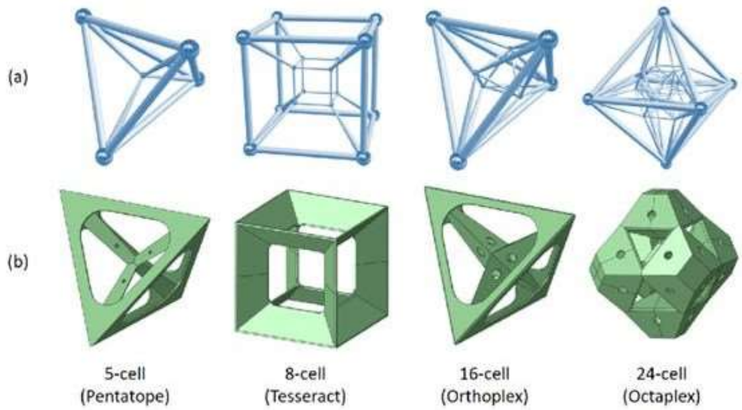


Figure 4: (a) Schlegel wire-frame diagrams of 4-dimensional regular convex polytopes (4-polytopes) and (b) 3D projected 4-polytope thin-walled unit cells based on the wire-frame diagrams.

3.1.3 SEMIREGULAR POLYTOPE

A semiregular polytope is a convex polytope where all faces are regular polygons but not all vertices have the same number of edges meeting at them. Examples include the Archimedean solids in three dimensions, and the 120-cell and 600-cell in four dimensions The term semiregular is often used to refer specifically to three-dimensional polyhedra, and the generalization to higher dimensions is often described using the term uniform polytopes.

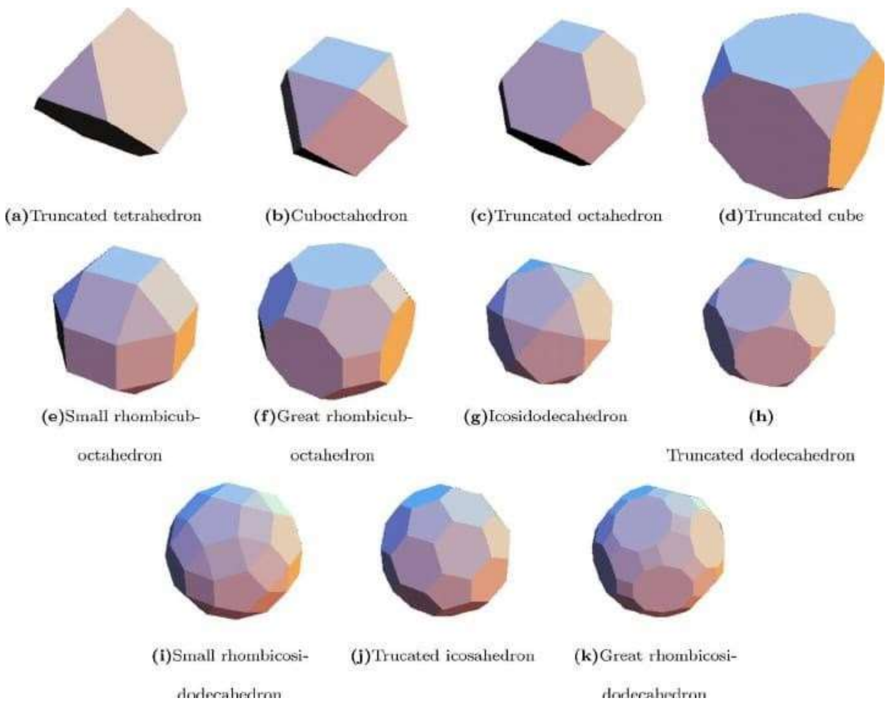


Figure 5: Archimedean solids

3.1.4 PROPERTIES

Combinatorial properties: faces (vertices, edges, ..., facets) of polytopes and their relations, with special treatments of the classes of low-dimensional polytopes and of polytopes "with few vertices;"

For a polytope P:

the faces of dimension 0 are the vertices of P.

the faces of dimension 1 are the edges of P.

the faces of dimension dim P−1 are the facets.

the empty set ∅ and P itself are called the improper faces, all other faces are proper.

Geometric properties: volume and surface area, mixed volumes, and quermassintegrals, including explicit formulas for the cases of the regular simplices, cubes, and cross-polytopes.

3.1.5 DUALITY

Two polytopes are said to be dual (and each is said to be a dual of the other) if their face-lattices are anti-isomorphic.

We note that when P and Q_1 are dual, then P and Q_2 are also dual if and only if Q_1 and Q_2 are equivalent.

Eg: The dual polytope of the 16-cell(4-orthoplex) is the tesseract (4-cube). The cells of the 16-cell are dual to the 16 vertices of the tesseract.

V-Polytopes and H-Polytopes

Convex polytopes are also called convex V-polytopes. Here V stands for vertices.

Dually, a convex H-polyhedron is the intersection of finitely many

closed halfspaces. A bounded convex H-polyhedron is called a convex H-polytope.

3.1.6 SIMPLEX

The unit simplex comes from a class of general polyhedra called simplex, having

vertex-description: given $n \geq k$

$$\text{conv}\{xl \in R^n \mid l = 1 \ldots k+1 \text{ , dim aff}\{xl\} = k\}$$

So defined, a simplex is a closed bounded convex set possibly not full-dimensional.

Examples of simplex, by increasing affine dimension, are: a point, any line segment, any triangle and its relative interior, a general tetrahedron, any five-vertex polychoron, and so on.

3.1.7 EULER'S FORMULA

$\upsilon - e + f = 2$ for every convex polyhedron, where υ, e, and f are the numbers of vertices, edges, and faces of the polyhedron.

3.1.8 HALFSPACE

A half-space is a set defined by a single affine inequality. Precisely, a half-space in $\mathbf{R}^n$ is a set of the form

$$\mathbf{H} = \left\{x \ : \ a^T x \leq b\right\},$$

where $a \in \mathbf{R}^n$, $b \in \mathbf{R}$. A half-space is a convex set, the boundary of which is a hyperplane.

A half-space separates the whole space in two halves. The complement of the half-space is the open half-space $\{x \,:\, a^T x > b\}$.

3.1.9 HYPERPLANE

A two-dimensional affine subset is called a plane. An n−1-dimensional affine subset of R^n is called a hyperplane. Every hyperplane partially bounds a halfspace.

3.2 CONVEX HULL

3.2.1 CONVEX HULL

The convex hull of a shape is the smallest convex set that contains the set. The convex hull can also be defined as the intersection of all convex sets which contain a given subset of a Euclidean space. Convex hull can also be defined as the set of all convex combinations of points in the subset. the convex hull may be visualized for a bounded subset of the plane, as the shape enclosed by a rubber band that is stretched around the subset.

Convex hulls of open sets are open, and convex hulls of compact sets are compact. Every compact convex set is the convex hull of its extreme points. The algorithm problems of finding the convex hull of a finite set of points in the plane or other low-dimensional Euclidean spaces are fundamental problems of computational geometry.

A convex hull is a geometric object, a polygon, that encloses all of those points in a given a set of points on a 2-dimensional plane. The vertices of this polygon maximize the area while minimizing the circumference.

ALGORITHMS IN CALCULATING CONVEX HULL

The Convex Hull has a wide variety of applications, ranging from image recognition in robotics to determining animal's home range in ethology. There are two algorithms in computing convex hull given below.

3.2.2 GRAHAM SCAN

Graham scan first sorts the points and then applies a linear-time scanning algorithm to finish building the hull. Graham scan is started by first finding the leftmost point l. Then we Sort out the points in counterclockwise order around l. To compare two points p and q, we check whether the triple l, p, q is oriented clockwise or counterclockwise. Once the points are sorted, we connected them in counterclockwise order, starting and ending at l. The result is a simple polygon with n vertices.

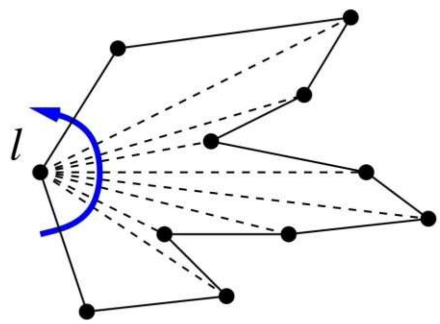


Figure 6: Graham scan

To change this polygon into the convex hull, now we apply the 'three-penny algorithm'. Consider we have three pennies, which will sit on three consecutive vertices p, q,r of the polygon.At the initial stages these are l and the two vertices after$l$. We now apply the following two rules over and over until a penny is moved forward onto l:

- If p, q, r are in counterclockwise order, move the back penny forward to the successor of r.
- If p, q, r are in clockwise order, remove q from the polygon, add the edge pr, and move the middle penny backward to the predecessor of p.

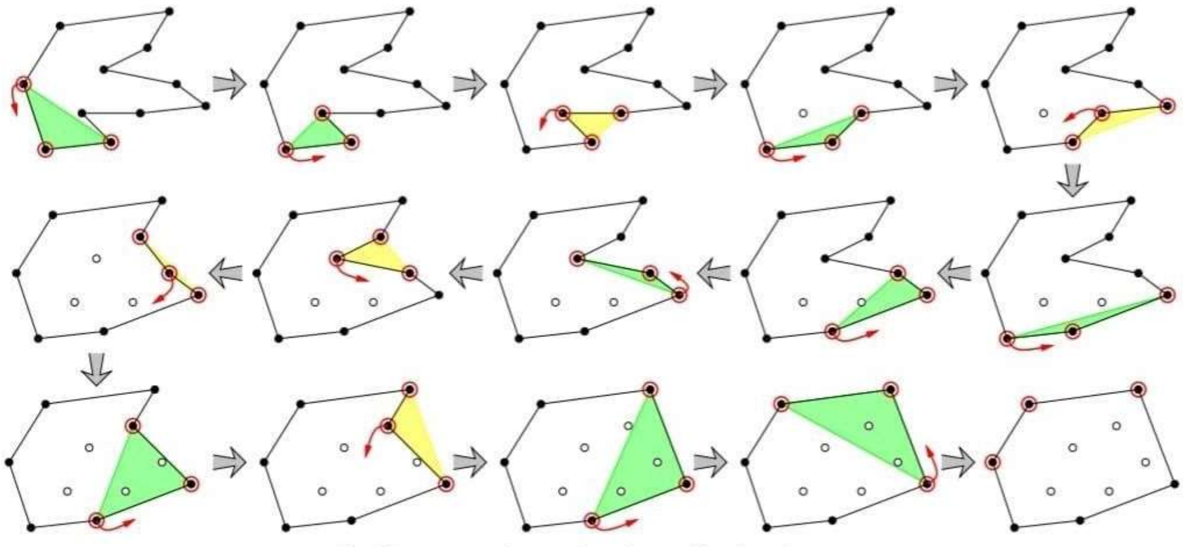


Figure 7: Graham scan method

Whenever a penny moves forward, it moves onto a vertex that hasn't seen a penny before.
Thus the first rule is applied n−2 times.
Also whenever a penny moves backwards, a vertex is removed from the polygon thus the second rule is applied exactly n − h times, where h is the number of convex hull vertices.

3.2.3 JARVIS'S ALGORITHM

The simplest algorithm for computing convex hulls simply simulates the process of wrapping a piece of string around the points. This algorithm is usually called Jarvis's march, it is also referred to as the gift-wrapping algorithm.
Jarvis's march starts by computing the leftmost point l (this is the point whose x-coordinate is smallest), as it is the leftmost point which must be the convex hull vertex.

Instead of sorting, it just loops through all of the points again in a brute force way to find the point that makes the smallest counterclockwise angle with reference to the previous vertex. It simply repeats this iteration through all of the points until all of the vertices are determined and it gets back to the starting point.

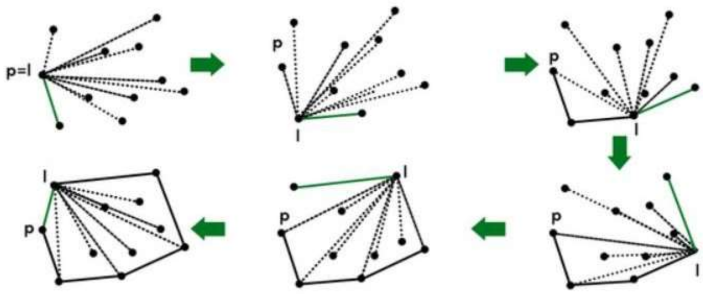


Figure 8: Jarvis Loop

Looping through every point for each vertex may seem a lot more inefficient, but the algorithm terminates as soon as it finds all of the vertices. This means that if the number of vertices is small, then it'll perform better than the Graham Scan algorithm.

By the way, the function for finding the point with the smallest counterclockwise angle is exactly the same as the one used previously that makes use of the cross product. Since the vertices are collinear points, it is a little easier to pick the point that is furthest away distance wise, without needing to worry about the slope of the line.

3.2.4 CARATHEODORY'S THEOREM

Theorem: (Caratheodory's Theorem) Let X be a nonempty subset of R^n.

1. Every nonzero vector of cone(X) can be represented as a positive combination of linearly independent vectors from X.

2. Every vector from conv(X) can be represented as a convex combination of at most $n + 1$ vectors from X.

Proof.

1) Let $x \in$ cone(X) and x 6= 0. Suppose m is the smallest integer such that x is of the form$\sum_{i=1}^{m} \alpha_i x_i$, where $\alpha_i > 0$ and $x_i \in X$.

Suppose that x_i are not linearly independent. Therefore, there exist λ_i with at least one λ_i positive, such that$\sum_{i=1}^{m} \lambda_i x_i = 0.$

Consider γ, the largest γ such that $\alpha - \gamma\lambda_i \geq 0$ for all i.

Then $\sum_{i=1}^{m}(\alpha_i - \overline{\gamma}\lambda)x_i$ is a representation of x as a positive combination of less than m vectors, contradiction. Hence, x_i are linearly independent.

2) Consider $Y = \{(x,1) : x \in X\}$. Let $x \in \text{conv}(X)$. Then $x = \sum_{i=1}^{m}\alpha_i x_i$, where $\sum_{i=1}^{m}\alpha_i = 1$, so $(x,1) \in \text{cone}(Y)$.

By 1), $(x,1) = \sum_{i=1}^{l}\alpha_i'(x_i,1)$, where $\alpha_i > 0$. Also, $(x_1,1),\ldots,(x_l,1)$ are linearly independent vectors in R^{n+1}(at most n+1). Hence, $x = \sum_{i=1}^{l}\alpha_i' x_i$,

$\sum_{i=1}^{m}\alpha_i' = 1$ □

Carathéodory's theorem serves as a cornerstone in convex geometry, elucidating a crucial relationship between the convex hull of a set of points and the dimensionality of the space in which those points reside. It states that any point within the convex hull of a set S in R^d can be expressed as a convex combination of at most d+1 points from S. This theorem not only characterizes the structure of convex hulls by limiting the number of extreme points necessary to define any point within them but also facilitates computational efficiency in constructing convex hulls by minimizing the number of extreme points to consider. Furthermore, Carathéodory's theorem underscores the intrinsic connection between the dimensionality of the space and the complexity of the convex hull, providing fundamental insights into the geometrical properties and computational aspects of convex sets.

4

APPLICATION OF CONVEX SET

Convex sets have wide-ranging applications in various fields due to their mathematical properties and geometric characteristics. Here are some common applications of convex sets

4.1. CONVEX OPTIMIZATION:

Convex optimization problems are widely studied and applied in fields such as engineering, economics, finance, and machine learning. Many real-world optimization problems can be cast as convex optimization problems, allowing for efficient and guaranteed algorithms to find optimal solutions.

Convex optimization is a field of mathematical optimization focused on solving problems where both the objective function and the constraints are convex. Convexity is a mathematical property that has important implications in optimization because it guarantees the existence of global optima and makes it possible to efficiently find solutions.

In convex optimization, the goal is to minimize (or maximize) a convex objective function subject to convex constraints. The general form of a convex optimization problem is;

Minimize f_0 (x)

Subject to f_i (x) ≤ 0, i = 1,…….,m

h_i (x) = 0, i= 1,. ...... ,p

x∈R^n is the optimization variable

f_0: R^n → R is the objective or cost function

f_i:R^n→R,i=1, . . . , m, are the inequality constraint functions

h_i: R^n → R , are the equality constraint functions some key points about convex optimization:

1. Global Optimality: Convex optimization problems have the property that any local minimum is also a global minimum. This is a significant advantage when compared to non-convex optimization problems.

2. Efficiency: Due to the structure of convex functions, there are efficient algorithms for solving convex optimization problems. Interior-point methods and first-order methods are commonly used.

3. Applications: Convex optimization has numerous applications, including but not limited to machine learning, finance, signal processing, control systems, and operations research. Many real-world problems can be formulated as convex optimization problems.

4. Duality: Convex optimization problems often come with associated dual problems. The duality theory provides insights into the optimization problem and helps in deriving bounds on the optimal value.

5. Examples: Common examples of convex optimization problems include linear programming, quadratic programming, and convex quadratic programming.

Overall, convex optimization is a powerful and widely used tool in various fields for solving optimization problems with desirable properties .

4.2. ECONOMICS AND GAME THEORY:

Convex sets hold immense importance in both economics and game theory due to their versatility and applicability across various contexts. In economics, they provide a fundamental framework for modeling individual behavior and economic interactions. For instance, utility functions, which represent individual preferences, exhibit convexity to capture diminishing marginal utility, a key concept in consumer theory. Similarly, the production possibility frontier, illustrating the trade-offs between different goods an economy can produce, relies on convex sets to depict increasing opportunity costs. Decision-making processes, including consumption choices and resource allocation, are analyzed using convex sets to represent feasible choice sets, aiding in understanding how individuals and firms make decisions under constraints.

In game theory, convex sets play a pivotal role in modeling strategic interactions among rational decision-makers. Strategies and their spaces are often represented as convex sets, ensuring that any convex combination of strategies remains within the feasible space. This property is essential for defining equilibrium concepts such as Nash equilibrium, where players' strategies are best responses to each other within a convex strategy space. Additionally, convex hulls, derived from convex sets, are used to identify equilibrium outcomes in various games, providing geometric interpretations that facilitate analysis and prediction.

Moreover, convex sets contribute to establishing the existence and uniqueness of equilibrium solutions in economic models, thereby informing discussions on market efficiency, welfare analysis, and economic dynamics. Furthermore, in decision-making under risk and uncertainty, convex sets provide a structured approach for representing feasible outcomes or states of nature, aiding in portfolio optimization and risk management strategies.

Overall, the importance of convex sets in economics and game theory lies in their ability to provide a rigorous mathematical framework for analyzing complex economic and strategic interactions, guiding decision-making processes, and facilitating the understanding of equilibrium outcomes and market dynamics. Their versatility and applicability make them indispensable tools for researchers, policymakers, and practitioners in these fields.

These applications demonstrate the versatility and importance of convex sets in solving a wide range of real-world problems across different disciplines.

5

ADVANCEMENT IN CONVEX GEOMETRY

5.1 MINKOWSKI THEOREM:

Minkowski's theorem is a fundamental result in convex geometry named after the German mathematician Hermann Minkowski. The theorem provides a deep connection between the geometry of convex sets and their properties related to integer points within them. There are different versions of Minkowski's theorem, and one of the most famous ones is the Minkowski Convex Body Theorem

5.2 GENERALISED MINKOWSKI :

Let K be a convex body in n-dimensional Euclidean space R^n, and let vol(K) denote its volume (in the sense of Lebesgue measure). If vol(K)>2^n then K contains at least one non-zero integer point.

In other words, if the volume of a convex body is sufficiently large (larger than 2^n), then there must be at least one non-zero integer point inside the convex body. This theorem highlights a profound connection between convex geometry and number theory.

This result has various applications in number theory, diophantine approximation, and lattice point problems. It establishes a link between geometric properties of convex bodies and the existence of integer points within these bodies.

Generalized Minkowski Theorem:

There are also more general versions of Minkowski's theorem that deal with lattices and convex sets. One such generalization is:

Let *L* be a lattice in R^n, and let *K* be a convex body that is centrally symmetric (symmetric with respect to the origin) and satisfies vol(K)>2^ndet (L). Then, *K* contains at least one lattice point other than the origin.

This version extends the theorem to consider lattices and provides conditions under which a convex body containing the origin must also contain a non-zero lattice point.

Minkowski's theorem and its generalizations have deep implications in various branches of mathematics, including convex geometry, number theory, and algebraic geometry. They are essential tools for understanding the interplay between the geometry of convex bodies and the distribution of lattice points within them.

References

- Dattorro, J (2004) Convex Optimization & Euclidean Distance Geometry.
- Jeffe Erickson (2002) CS 373. Non-Lecture E: Convex Hulls. Fall 2002. E Convex Hulls
- Dimitri Bertsekas, Angelia Nedic, Asuman Ozdaglar (2003) Convex Analysis and Optimization.
- Dimitri P Bertsekas (2009). *Convex Optimization Theory*.
- Gruber, P. M. (2007) Convex and discrete geometry.
- Gabrielis Cerniauskas & Alam, Parvez (2023). Cubically symmetric mechanical metamaterials projected from 4th dimensional geometries reveal high specific properties in shear.
- Jonathan Borwein, Adrian Lewis (2000). Convex Analysis and Nonlinear Optimization: Theory and Examples, Second Edition
- Hiriart-Urruty, Jean-Baptiste, and Lemaréchal, Claude (2004). *Fundamentals of Convex analysis*.
- Hiriart-Urruty, Jean-Baptiste; Lemaréchal, Claude (1993). *Convex analysis and minimization algorithms, Volume I: Fundamentals*.