Project Report

On

RHOTRIX THEORY

Submitted

in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE

in

MATHEMATICS

by

CHAITANYA KRISHNA

(Register No. SM22MAT006)

(2023-2024)

Under the Supervision of

MS. NISHA OOMMEN

DEPARTMENT OF MATHEMATICS

ST. TERESA'S COLLEGE (AUTONOMOUS)

ERNAKULAM, KOCHI - 682011

APRIL 2024

ST. TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM

CERTIFICATE

This is to certify that the dissertation entitled, **RHOTRIX THEORY** is a bonafide record of the work done by Ms. **CHAITANYA KRISHNA** under my guidance as partial fulfillment of the award of the degree of **Master of Science in Mathematics** at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

Date: 29/04/24
Place: Ernakulam

Nisha Oommen
Assistant Professor and HOD,
Department of Mathematics and Statistics (SF),
St. Teresa's College(Autonomous),
Ernakulam.

Nisha Oommen
Assistant Professor and HOD,
Department of Mathematics and Statistics (SF),
St. Teresa's College(Autonomous),
Ernakulam.

External Examiners

1: Dr. Lejo J. Manavalan

2: Dr. Parameswaran R.

DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of **MS. NISHA OOMMEN**, Assistant Professor, Department of Mathematics, St. Teresa's College(Autonomous), Ernakulam and has not been included in any other project submitted previously for the award of any degree.

Ernakulam.

Date: 29/04/24

CHAITANYA KRISHNA

SM22MAT006

ACKNOWLEDGEMENTS

The effort taken by the completion of this project would not have been possible without the kind of support and help rendered by many individuals. I would like to acknowledge my sincere thanks to all of them.

First, I would like to thank God Almighty for His glorious blessings which have been accompanied throughout the course of my work. I firmly believe, it was with His help that I was able to complete this project successfully.

I extend my deepest sense of gratitude to my supervising faculty Ms. Nisha Oommen, for her patience and support together with proper guidance throughout the completion of my project. I record my sincere thanks to Dr. Ursala Paul, Head of the Department for giving me such attention and time.

I am highly indebted to all the teaching and non-teaching staff of the Department of Mathematics and Statistics, without whom, the project will be incomplete.

In addition, I also thank all my classmates, whose vibrant support had been with me throughout the course of completion of this project.

I also express my gratitude to the college librarian of St. Teresa's College (Autonomous) for providing various facilities for this project.

Ernakulam. CHAITANYA KRISHNA

Date: 29/04/24 **SM22MAT006**

ST.TERESA'S COLLEGE (AUTONOMOUS) ERNAKULAM

Certificate of Plagiarism Check for Dissertation

Author Name	CHAITANYA KRISHNA
Course of Study	M.Sc. Mathematics
Name of Guide	Ms. NISHA OOMMEN
Department	POST GRADUATE MATHEMATICS
Acceptable Maximum Limit	20%
Submitted By	library@teresas.ac.in
Paper Title	RHOTRIX THEORY
Similarity	4% AI 0%
Paper ID	1542577
Submission Date	2024-03-18 14:31:05

Signature of Student

Signature of Guide

Checked By
College Librarian

* This report has been generated by DrillBit Anti-Plagiarism Software

Contents

Introduction

Rhotrix theory is a relatively new area of study in linear mathematical algebra, which was introduced by A. O. Ajibade in 2003.It deals with arrays in a rhomboidal structure, unlike matrix theory, which deals with arrays in rectangular form. This introduction was inspired by the discussion of K. T. Atanassov and A. G. Shannon on the possibility of existence of mathematical arrays that are, in some way, between two-dimensional vectors and 2×2 matrices, in 1998. As an extension to this, Ajibade introduced a mathematical object that is in some way between 2×2 matrices and 3×3 matrices, and named it as a three-dimensional rhotrix. Rhotrix theory has been able to pique the interest of various researchers around the globe since its introduction in 2003. Development and expansion of these concepts have shown that rhotrix theory has introduced a new paradigm of matrix theory, which attracted researchers for its mathematical enrichment. Apart from Ajibade, B.Sani, A. Aminu, Abdul Mohammed, A.O. Isere, M. Balarabe are other major contributors of developing rhotrix theory into its current form.

It is germane to mention the existence of two methods of multiplication of rhotrices in literature currently. The first one is based on the heart of the rhotrix, called the heart-oriented multiplication, which was introduced by Ajibade. The other one was introduced by B. Sani, called the row-column multiplication. Based on these two methods, rhotrix theory can be classified into two classes, namely commutative rhotrix theory and non-commutative rhotrix theory. Commutative rhotrix theory is based on heart-oriented multiplication, while the latter is based on row-column multiplication. These classes are named on the basis of heart-oriented multiplication being a commutative operation, while row-column multiplication being non-commutative.

This paper comprises a total of three chapters. The first chapter deals with an introduction to rhotrices, including basic definitions related to the structure of a rhotrix, and representations of rhotrices. The second chapter contains operations on rhotrices, such as addition of two rhotrices, multiplication of a rhotrix with a scalar, the two methods of multiplication of two rhotrices, rhotrix exponent rule and some results regarding them.

In the third chapter, we see an application of rhotrices on cryptography. We will see two different methods of encryption and decryption, the commutative and the non-commutative methods. These methods are inspired by the existence of encryption and decryption methods using matrices. These methods are equally secure to the method of using matrices, if not more.

Chapter 0

Preliminaries

0.1 Group Theory

Definition 0.1.1 A *group* $\langle G, * \rangle$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

G1 For all $a, b, c \in G$, we have

$$(a * b) * c = a * (b * c). \quad \textbf{(associativity of *)}$$

G2 There is an element e in G such that for all $x \in G$,

$$e * x = x * e = x. \quad \textbf{(identity element e for *)}$$

G3 Corresponding to each $a \in G$, there is an element a' in G such that

$$a * a' = a' * a = e. \quad \textbf{(inverse a' of a)}$$

Definition 0.1.2 A group G is *abelian* if its binary operation is commutative

0.2 Vector Space

Definition 0.2.1 A *vector space* (or linear space) consists of the following:

1. a field F of scalars
2. a set V of objects, called vectors
3. a rule (or operation), called vector addition, which associates with each pair of vectors α, β in V a vector $\alpha + \beta$ in V, called the sum of α and β, in such a way that
   (a) addition is commutative, $\alpha + \beta = \beta + \alpha$
   (b) addition is associative, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
   (c) there is a unique vector 0 in V, called the zero vector, such that $\alpha + 0 = \alpha$, for all α in V
   (d) for each vector α in V there is a unique vector $-\alpha$ in V such that $\alpha + (-\alpha) = 0$
4. a rule (or operation), called scalar multiplication, which associates with each scalar c in F and vector α in V, a vector $c\alpha$ in V, called the product of c and α, in such a way that
   (a) $1\alpha = \alpha$ for every α in V
   (b) $(c_1c_2)\alpha = c_1(c_2\alpha)$
   (c) $c(\alpha + \beta) = c\alpha + c\beta$
   (d) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$

0.3 Cryptography

Cryptography is technique of securing information and communications through use of codes so that only those person for whom the information is intended can understand it and process it. Thus preventing unauthorized access to information. The prefix "crypt" means "hidden" and suffix "graphy" means "writing". In Cryptography the techniques which are use to protect information are obtained from mathematical concepts and a set of rule based calculations known as algorithms to convert messages in ways that make it hard to decode it. These algorithms are used for cryptographic key generation, digital signing, verification to protect data privacy, web browsing on internet and to protect confidential transactions such as credit card and debit card transactions.

Cryptography is often associated with the process where an ordinary plain text is converted to cipher text which is the text made such that intended receiver of the text can only decode it and hence this process is known as encryption. The process of conversion of cipher text to plain text this is known as decryption.

Definition 0.3.1 *Symmetric Key cryptography* is an encryption system where the sender and receiver of message use a single common key to encrypt and decrypt messages. Symmetric Key Systems are faster and simpler but the problem is that sender and receiver have to somehow exchange key in a secure manner.

Definition 0.3.2 *Hill cipher* is a polygraphic substitution cipher based on linear algebra.Each letter is represented by a number modulo 26. Often the simple scheme $A = 0$, $B = 1$, $\ldots$, $Z = 25$ is used, but this is not an essential feature of the cipher. To encrypt a message, each block of n letters (considered as an n-component vector) is multiplied by an invertible $n \times n$ matrix, against modulus 26. To decrypt the message, each block is multiplied by the inverse of the matrix used for encryption.
The matrix used for encryption is the cipher key, and it should be chosen randomly from the set of invertible $n \times n$ matrices (modulo 26).

Chapter 1

BASIC CONCEPTS OF RHOTRIX

A rhorix R_n is a rhomboidal arrangement of entries with odd dimension $n \in 2\mathbb{Z}^{+}+1$ defined by

$$R_n = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{31} & a_{22} & a_{13} & & \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ a_{n1} & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & a_{1n} \\ & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ & & a_{nn-2} & a_{n-1n-1} & a_{n-2n} & & \\ & & & a_{nn} & & & \end{array} \right\rangle$$

The smallest possible dimension of a rhotrix is 3. A three-dimensional rhotrix in its general form is given by

$$R_3 = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \mathbb{R} \right\}$$

1.1 Structure of a Rhotrix

Definition 1.1.1 Every rhotrix has two axes -- a horizontal axis and a vertical axis. The horizontal axis of a rhotrix is the array of entries running from the left to the right-hand side of the rhotrix, while the vertical axis is an array of entries running from the top to the bottom of the rhorix. Every rhotrix has a major horizontal and a major vertical axis.

Example 1.1.1. Consider R_3. The horizontal axis of R_3 is the set of values 'b, c, d' and the vertical axis is the set of values 'a, c, e'.

Definition 1.1.2 The element at the intersection of the two axes of a rhorix R is called the heart of R denoted by $h(R)$.

Example 1.1.2. Consider R_3. The heart of R_3 is the element c. Thus, R_3 can be rewritten as

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \tag{1.1}$$

.

Definition 1.1.3 The entries at the four corners of a rhotrix are called the vertices of the rhotrix.

Example 1.1.3. Consider R_3. Then the vertices of R_3 are a, b, d and e.

Example 1.1.4. Consider the rhotrix

$$R_5 = \left\langle \begin{array}{ccccc} & & a & & \\ & f & d & b & \\ k & i & g & e & c \\ & l & j & h & \\ & & m & & \end{array} \right\rangle.$$

The vertices of R_5 are a, k, c and m.

Definition 1.1.4 The rows of a rhotrix are the arrays of entries running from the top-left to the bottom-right side of the rhotrix The columns of a rhotrix are the arrays of entries running from the top-right to the bottom-left side of the rhotrix.

Example 1.1.5. Consider R_3. The first row consists of the elements a and d, the second row consists of c, and the third row consists of the elements b and e. The first column consists of elements a and b, the second column consists of c, and the third column consists of elements d and e.

Definition 1.1.5 The number of entries in the axes of a rhotrix is called the dimension of the rhotrix. By definition of the structure of a rhotrix, the number of elements in the horizontal and the vertical axes are equal. All rhotrices are of odd dimension, with the dimension at least 3. The dimension is usually indicated by attaching it as a subscript to the variable used to represent it.

Example 1.1.6. The dimension of R_3 is 3, and that of R_5 is 5.

Definition 1.1.6 The number of elements in a rhotrix is called its cardinality. The cardinality of a rhotrix of dimension n is given by $\frac{n^2+1}{2}$, where $n \in 2\mathbb{Z}^+ + 1$.

Example 1.1.7. R_3 has a dimension 3 and thus cardinality of R_3 is $\frac{3^2+1}{2} = 5$.

Example 1.1.8. R_5 is of dimension 5 and therefore has a cardinality $\frac{5^2+1}{2} = 13$.

Definition 1.1.7 Two rhotrices of same dimension are said to be equal if the elements in the corresponding positions are equal.

Example 1.1.9. Consider the rhotrices $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q =$

$\left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$. Then R and Q are said to be equal if $a = f, b = g, h(R) = h(Q), d = j$ and $e = k$, and we denote it by $R = Q$.

Definition 1.1.8 A rhotrix whose all elements are 0 is called a zero rhotrix. A zero rhotrix of dimension 3 is given by $\left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$.

Definition 1.1.9 A rhotrix in which the value of its heart is 1 called a unit heart rhotrix.

Example 1.1.10. The rhotrix $\left\langle \begin{array}{ccc} & 3 & \\ 6 & 1 & 0 \\ & 13 & \end{array} \right\rangle$ is a three-dimensional unit heart rhotrix.

Definition 1.1.10 The determinant of a three-dimensional rhotrix $R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ is given by $det(R) = c(ae - bd)$.

Remark 1.1.1. The determinant of rhotrices of higher dimension $n, n \in 2\mathbb{Z}^+ + 1$ are yet to be determined.

Definition 1.1.11 The rhotrix obtained by interchanging the row and column vectors of the given rhotrix R_n is called the transpose of R_n and is denoted by R_n^T.

Example 1.1.11. The transpose of the rhotrix $\left\langle \begin{array}{ccc} & 3 & \\ 2 & 1 & 0 \\ & 6 & \end{array} \right\rangle$ is $\left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 2 \\ & 6 & \end{array} \right\rangle$.

Definition 1.1.12 The rhotrix R is called a symmetric rhotrix if $R^T = R$ and a skew-symmetric rhotrix if $R^T = -R$.

Example 1.1.12. Consider the rhotrix $R = \left\langle \begin{matrix} & 3 & \\ 4 & 6 & 4 \\ & 2 & \end{matrix} \right\rangle$. Its transpose is $R^T = \left\langle \begin{matrix} & 2 & \\ 4 & 6 & 4 \\ & 3 & \end{matrix} \right\rangle = R$. Thus R is a symmetric rhotrix.

Now consider the rhotrix $S = \left\langle \begin{matrix} & 0 & \\ 4 & 0 & -4 \\ & 0 & \end{matrix} \right\rangle$. Its transpose is $S^T = \left\langle \begin{matrix} & 0 & \\ -4 & 0 & 4 \\ & 0 & \end{matrix} \right\rangle = -S$. Thus S is a skew-symmetric rhotrix.

1.2 Representation of Arbitrary Rhotrices

An arbitrary rhotrix in its general form can be represented in mainly two ways, namely Single Index Method and Row-Column Method.

1.2.1 Single Index Method

In this method the indices used are natural numbers from 1 to $\frac{n^2+1}{2}$, where n is the dimension of the rhotrix. This can be done in two ways. One, by indicating row-wise, as in the following

$$R_5 = \left\langle \begin{matrix} & & a_1 & & \\ & a_6 & a_4 & a_2 & \\ a_{11} & a_9 & a_7 & a_5 & a_3 \\ & a_{12} & a_{10} & a_8 & \\ & & a_{13} & & \end{matrix} \right\rangle.$$

The other method is by allowing the indices to rum horizontally, from left to right, as in the following

$$R_5 = \left\langle \begin{array}{ccccc} & & a_1 & & \\ & a_2 & a_3 & a_4 & \\ a_5 & a_6 & a_7 & a_8 & a_9 \\ & a_{10} & a_{11} & a_{12} & \\ & & a_{13} & & \end{array} \right\rangle.$$

1.2.2 Row-Column Method

Two indices are used in this method, the first one indicating the row in which the entry lies and the second one indicating the column in which the entry lies. Thus we have a indexing of the form

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{31} & a_{22} & a_{13} & \\ a_{51} & a_{42} & a_{33} & a_{24} & a_{15} \\ & a_{53} & a_{44} & a_{35} & \\ & & a_{55} & & \end{array} \right\rangle.$$

Chapter 2

OPERATIONS ON RHOTRICES

2.1 Addition of Rhotrices

Two rhotrices can be added only if they have the same dimension.

Consider the three-dimensional rhotrices R and Q, where $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$. The operation addition(+) on R and Q is defined by

$$R+Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle. \tag{2.1}$$

This can be extended to rhotrices of dimensions greater than 3, where the element on a particular position on the sum rhotrix is obtained by finding the sum of the elements in the corresponding positions of the two rhotrices.

Example 2.1.1. $\left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 9 \\ & 2 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & 5 & \\ 7 & 3 & 2 \\ & 10 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 9 & \\ 7 & 4 & 11 \\ & 12 & \end{array} \right\rangle$

Result 2.1.1. The set of all n-dimensional rhotrices forms an abelian group under the operation addition (+).

Proof:

(i) The sum of two rhotrices is again a rhotrix of the same dimension and thus, the set of rhotrices is closed under addition.

(ii) Consider the rhotrices $P = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle$, $Q = \left\langle \begin{matrix} & f & \\ g & i & j \\ & k & \end{matrix} \right\rangle$ and $R = \left\langle \begin{matrix} & l & \\ m & n & o \\ & p & \end{matrix} \right\rangle$.

$$(P+Q)+R = \left(\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & i & j \\ & k & \end{matrix} \right\rangle \right) + \left\langle \begin{matrix} & l & \\ m & n & o \\ & p & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & a+f & \\ b+g & c+i & d+j \\ & e+k & \end{matrix} \right\rangle + \left\langle \begin{matrix} & l & \\ m & n & o \\ & p & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & (a+f)+l & \\ (b+g)+m & (c+i)+n & (d+j)+o \\ & (e+k)+p & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & a+(f+l) & \\ b+(g+m) & c+(i+n) & d+(j+o) \\ & e+(k+p) & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f+l & \\ g+m & i+n & j+o \\ & k+p & \end{matrix} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle + \left(\left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & l & \\ m & n & o \\ & p & \end{array} \right\rangle \right) = P + (Q + R)$$

Thus, the set of rhotrices is associative under the operation addition.

(iii) Let $0 = \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle$ be the identity rhotrix under addition.

Consider the rhotrix $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$. Then $P + 0 = P$ and $0 + P = P$.

$$\Longrightarrow \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

$$\Longrightarrow \left\langle \begin{array}{ccc} & a+p & \\ b+q & c+r & d+s \\ & e+t & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & p+a & \\ q+b & r+c & s+d \\ & t+e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

From both these equations, we get $p = 0, q = 0, r = 0, s = 0$ and $t = 0$. And therefore,

$$0 = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle,$$

the zero rhotrix, is the identity rhotrix.

(iv) Consider the rhotrices $P = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle$ and $-P = \left\langle \begin{matrix} & -a & \\ -b & -c & -d \\ & -e & \end{matrix} \right\rangle$.

Then,

$$P + (-P) = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & -a & \\ -b & -c & -d \\ & -e & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & a+(-a) & \\ b+(-b) & c+(-c) & d+(-d) \\ & e+(-e) & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$$

And,

$$(-P) + P = \left\langle \begin{matrix} & -a & \\ -b & -c & -d \\ & -e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle$$

$$\left\langle \begin{matrix} & (-a)+a & \\ (-b)+b & (-c)+c & (-d)+d \\ & (-e)+e & \end{matrix} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle = 0$$

Thus, $-P$ is the inverse of P under addition.

(v) Consider P and Q as defined above.

$$P + Q = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a+f & \\ b+g & c+i & d+j \\ & e+k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & f+a & \\ g+b & i+c & j+d \\ & k+e & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = Q + P$$

Thus, rhotrices are commutative under addition.

From (i), (ii), (iii), (iv) and (v), we can conclude that the set of all three-dimensional rhotrices form an abelian group under the operation addition.

Since the operation addition can be generalized into higher dimensional rhotrices, the result is true for the set of all rhotrices and hence the result.

2.2 Scalar Multiplication

Scalar multiplication on a rhotrix is defined by multiplying each element of the rhotrix by that scalar.

Consider the rhotrix $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and the scalar $\alpha \in \mathbb{R}$. Then αR is given by

$$\alpha R = \alpha \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \alpha a & \\ \alpha b & \alpha h(R) & \alpha d \\ & \alpha e & \end{array} \right\rangle \tag{2.2}$$

Example 2.2.1. Let $R = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 9 \\ & 2 & \end{array} \right\rangle$. Then $3R = 3\left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 9 \\ & 2 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 12 & \\ 0 & 3 & 27 \\ & 6 & \end{array} \right\rangle$

Theorem 2.2.1 *Let $\hat{R}_3$ be the set of all three-dimensional rhotrices. Then $\hat{R}_3$ is a vector space over $\mathbb{R}$, under standard addition and scalar multiplication.*

Proof: From the proof of Result 2.1.1, we have,

(i) $\hat{R}_3$ is closed under vector addition.

(ii) Vector addition is commutative.

(iii) Vector addition is associative.

(iv) There exists an additive identity.

(v) There exists an additive inverse.

Now, consider the rhotrices $P = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle$ and $Q = \left\langle \begin{matrix} & f & \\ g & i & j \\ & k & \end{matrix} \right\rangle$ and scalars m and n.

(vi) $mP = m\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & ma & \\ mb & mc & md \\ & me & \end{matrix} \right\rangle$. Clearly, $mP \in \hat{R}_3$.

(vii)

$$1 \cdot P = \left\langle \begin{matrix} & 1 \cdot a & \\ 1 \cdot b & 1 \cdot c & 1 \cdot d \\ & 1 \cdot e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle = P$$

(viii)

$$m(P+Q) = m\left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & i & j \\ & k & \end{matrix} \right\rangle \right\}$$

$$= m\left\langle \begin{matrix} & a+f & \\ b+g & c+i & d+j \\ & e+k & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & m(a+f) & \\ m(b+g) & m(c+i) & m(d+j) \\ & m(e+k) & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & ma+mf & \\ mb+mg & mc+mi & md+mj \\ & me+mk & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & ma & \\ mb & mc & md \\ & me & \end{matrix} \right\rangle + \left\langle \begin{matrix} & mf & \\ mg & mi & mj \\ & mk & \end{matrix} \right\rangle$$

$$= m\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle + m\left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle = mP + mQ$$

(ix)

$$(m+n)P = (m+n)\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & (m+n)a & \\ (m+n)b & (m+n)c & (m+n)d \\ & (m+n)e & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ma+na & \\ mb+nb & mc+nc & md+nd \\ & me+ne & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ma & \\ mb & mc & md \\ & me & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & na & \\ nb & nc & nd \\ & ne & \end{array} \right\rangle$$

$$= m\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle + n\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = mP + nP$$

(x)

$$m(nP) = m\left\{ n\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \right\}$$

$$= m\left\langle \begin{array}{ccc} & na & \\ nb & nc & nd \\ & ne & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & m(na) & \\ m(nb) & m(nc) & m(nd) \\ & m(ne) & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & (mn)a & \\ (mn)b & (mn)c & (mn)d \\ & (mn)e & \end{array} \right\rangle$$

$$(mn) \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = (mn)P$$

Thus, $\hat{R}_3$ forms a vector space over $\mathbb{R}$

Result 2.2.1. $\hat{R}_3$ has dimension 5 and the vectors $I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, J = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, K = \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, L = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle, M = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle$ forms a basis for $\hat{R}_3$.

Proof: First we prove that the vectors I,J,K,L and M are linearly independent.

Consider scalars $c_1, c_2, c_3, c_4 and c_5$. Now,

$$c_1 I + c_2 J + c_3 K + c_4 L + c_5 M = 0$$

$$\Rightarrow c_1 \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle + c_2 \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle + c_3 \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle + c_4 \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle + c_5 \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle = \Bigg\langle$$

$$\Longrightarrow \left\langle \begin{array}{ccc} & c_1 & \\ c_2 & c_3 & c_4 \\ & c_5 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

Then by equality of rhotrices, we have,

$$c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0$$

Thus, we have, I, J, K, L and M are linearly independent.

Now, let $\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ be an arbitrary element in $\hat{R_3}$.

Then, clearly,

$$\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = c\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle + a\left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle + b\left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle + d\left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle + e\left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle$$

Thus, the vectors I, J, K, L and M spans $\hat{R_3}$.

And hence we have, I, J, K, L and M forms a basis for $\hat{R_3}$.

Consequently $\hat{R_3}$ is of dimension 5.

2.3 Rhotrix Multiplication

Rhotrix multiplication of two rhotrices of same dimension can be done in two ways, namely heart-oriented multiplication and row-column multiplication.

2.3.1 Heart-Oriented Multiplication

Consider two 3-dimensional rhotrices $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$.

Then their heart-oriented product $R \circ Q$ s given by

$$R \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ah(Q) + fh(R) & \\ bh(Q) + gh(R) & h(R)h(Q) & dh(Q) + jh(R) \\ & eh(Q) + kh(R) & \end{array} \right\rangle \tag{2.3}$$

Example 2.3.1.1. Let $R = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 3 \\ & 2 & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & 2 & \\ 6 & 3 & 2 \\ & 1 & \end{array} \right\rangle$.

Then,

$$R \circ Q = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 3 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 6 & 3 & 2 \\ & 1 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 4 \cdot 3 + 2 \cdot 1 & \\ 0 \cdot 3 + 6 \cdot 1 & 1 \cdot 3 & 3 \cdot 3 + 2 \cdot 1 \\ & 2 \cdot 3 + 1 \cdot 1 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 14 & \\ 6 & 3 & 11 \\ & 7 & \end{array} \right\rangle$$

This can be generalized into rhotrices of higher dimensions where the elements in the product rhotrix is obtained by multiplying the corresponding element of the first rhotrix by the heart of the second rhotrix and adding it to the product of the corresponding element of the second rhotrix and the heart of the first rhotrix, while the heart of the product rhotrix is obtained by multiplying the hearts of the rhotrices.

Result 2.3.1.1. Identity rhotrix under heart-oriented multiplication is given by

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle \tag{2.4}$$

Proof: Let $I = \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle$ be the identity rhotrix under heart-oriented multiplication.

Consider the rhotrix $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$. Then $P \circ I = I \circ P = P$

$$\Longrightarrow \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

$$\implies \left\langle \begin{array}{ccc} & ar+pc & \\ br+qc & cr & dr+sc \\ & er+tc & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & pc+ar & \\ qc+br & rc & sc+dr \\ & tc+er & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

$$\begin{aligned} \implies ar+pc &= 0 \\ br+qc &= 0 \\ cr &= 1 \\ dr+sc &= 0 \\ er+tc &= 0 \end{aligned}$$

Solving these five equations, we obtain $p = q = s = t = 0$ and $r = 1$

Thus,

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

is the identity rhotrix under heart-oriented multiplication of dimension 3.

Result 2.3.1.2. Inverse of a rhotrix under heart-oriented multiplication is given by

$$Q = P^{-1} = \frac{-1}{c^2} \left\langle \begin{array}{ccc} & a & \\ b & -c & d \\ & e & \end{array} \right\rangle \tag{2.5}$$

Proof: Let the rhotrix $Q = \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle$ be the inverse of the rhotrix $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$.

Then, $P \circ Q = Q \circ P = I$

$$\Longrightarrow \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

$$\Longrightarrow \left\langle \begin{array}{ccc} & ar+pc & \\ br+qc & cr & dr+sc \\ & er+tc & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & pc+ar & \\ qc+br & rc & sc+dr \\ & tc+er & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

$$\Longrightarrow ar + pc = 0$$
$$br + qc = 0$$
$$cr = 1$$
$$dr + sc = 0$$
$$er + tc = 0$$

Solving these five equations, we obtain,

$$r = \tfrac{1}{c}, p = \tfrac{-a}{c^2}, q = \tfrac{-b}{c^2}, s = \tfrac{-d}{c^2} \text{ and } t = \tfrac{-e}{c^2}.$$

Thus, we have,

$$Q = P^{-1} = \frac{-1}{c^2}\left\langle \begin{matrix} & a & \\ b & -c & d \\ & e & \end{matrix} \right\rangle \tag{2.6}$$

Theorem 2.3.1.1 *A necessary condition for a rhotrix P to be invertible is that $h(P) \neq 0$.*

Proof: If P is invertible, there exists a rhotrix P^{-1} such that $P \circ P^{-1} = I$

$$\Longrightarrow h(P \circ P^{-1}) = h(I)$$
$$\Longrightarrow h(P)h(P^{-1}) = 1$$
$$\Longrightarrow h(P^{-1}) = \frac{1}{h(P)}$$
$$\Longrightarrow h(P) \neq 0$$

Theorem 2.3.1.2 *For any rhotrix $R \neq 0$, $R^2 = 0$ iff $h(R) = 0$.*

Proof: Let $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$. Then,

$$R^2 = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 2ah(R) & \\ 2bh(R) & h(R)^2 & 2dh(R) \\ & 2eh(R) & \end{array} \right\rangle$$

$$= 2h(R) \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

Clearly,

$$R^2 = 0 \iff 2h(R) \cdot R = 0 \iff h(R) = 0$$

since $R \neq 0$

Theorem 2.3.1.3 *The set of all three-dimensional invertible rhotrices over $\mathbb{R}$ forms an abelian group under heart-oriented rhotrix multiplication.*

Proof: Let

$$R = \left\{ P = \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle : h(P) \neq 0, a, b, h(P), d, e \in R \right\}$$

(i) Let $P, Q \in \mathbb{R}$, where $P = \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$. Then,

$$P \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ah(Q) + fh(P) & \\ bh(Q) + gh(P) & h(P)h(Q) & dh(Q) + jh(P) \\ & eh(Q) + kh(P) & \end{array} \right\rangle$$

Since $h(P) \neq 0$ and $h(Q) \neq 0$, $h(P)h(Q) \neq 0$ and therefore, R is closed under heart-oriented multiplication.

(ii) Let $P, Q, T \in R$, where $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$, $Q = \left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle$ and $T = \left\langle \begin{array}{ccc} & l & \\ m & n & o \\ & p & \end{array} \right\rangle$.

$$(P \circ Q) \circ R = \left(\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle \right) \circ \left\langle \begin{array}{ccc} & l & \\ m & n & o \\ & p & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ai + fc & \\ bi + gc & ci & di + jc \\ & ei + kc & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & l & \\ m & n & o \\ & p & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & (a+f)+l & \\ (b+g)+m & (c+i)+n & (d+j)+o \\ & (e+k)+p & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & (ai+fc)n+lci & \\ (bi+gc)n+mci & cin & (di+jc)n+oci \\ & (ei+kc)n+pci & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ain+fcn+lci & \\ bin+gcn+mci & cin & din+jcn+oci \\ & ein+kcn+pci & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ain+(fn+li)c & \\ bin+(gn+mi)c & cin & din+(jn+oi)c \\ & ein+(kn+pi)c & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & fn+li & \\ gn+mi & in & jn+oi \\ & kn+pi & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left(\left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & l & \\ m & n & o \\ & p & \end{array} \right\rangle \right) = P \circ (Q \circ R)$$

Thus, R is associative under heart-oriented multiplication.

(iii) We know that, $I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$ is the identity rhotrix under heart-oriented multiplication.

(iv) If $P = \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle \in R$, then $P^{-1} = \frac{-1}{h(P)^2} \left\langle \begin{array}{ccc} & a & \\ b & -h(P) & d \\ & e & \end{array} \right\rangle$ is the inverse of P. Since $h(P) \neq 0, -h(P) \neq 0$

(v) Let $P, Q \in R$, where $P = \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$. Then,

$$P \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ah(Q) + fh(P) & \\ bh(Q) + gh(P) & h(P)h(Q) & dh(Q) + jh(P) \\ & eh(Q) + kh(P) & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & fh(P) + ah(Q) & \\ fh(P) + ah(Q) & h(Q)h(P) & jh(P) + dh(Q) \\ & kh(P) + eh(Q) & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle = Q \circ P$$

Thus, R is commutative under heart-oriented multiplication.

From (i), (ii), (iii), (iv) and (v), we can conclude that the set of all three-dimensional rhotrices form an abelian group under heart-oriented multiplication.

2.3.2 Row-Column Multiplication

Consider two 3-dimensional rhotrices $R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle$.

Then their heart-oriented product $R \circ Q$ s given by

$$R \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & i & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & ci & aj+dk \\ & bj+ek & \end{array} \right\rangle \tag{2.7}$$

Example 2.3.2.1. Let $R = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 3 \\ & 2 & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & 2 & \\ 6 & 3 & 2 \\ & 1 & \end{array} \right\rangle$.

Then,

$$R \circ Q = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 3 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 6 & 3 & 2 \\ & 1 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 4\cdot 2+3\cdot 6 & \\ 0\cdot 2+2\cdot 6 & 1\cdot 3 & 4\cdot 2+3\cdot 1 \\ & 0\cdot 2+2\cdot 1 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 26 & \\ 12 & 3 & 11 \\ & 2 & \end{array} \right\rangle$$

This can be generalized into rhotrices of higher dimensions where the elements in the product rhotrix is obtained by multiplying the corresponding row of the first rhotrix to the corresponding column of the second rhotrix.

Result 2.3.2.1. Identity rhotrix under heart-oriented multiplication is given by

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle \tag{2.8}$$

Proof: Let $I = \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle$ be the identity rhotrix under row-column multiplication.

Consider the rhotrix $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$. Then $P \circ I = I \circ P = P$

$$\Longrightarrow \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

$$\Longrightarrow \left\langle \begin{array}{ccc} & ap + dq & \\ bp + eq & cr & as + dt \\ & bs + et & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} & pa + sb & \\ qa + tb & rc & pd + se \\ & qd + te & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

$$\implies ap + dq = a$$
$$bp + eq = b$$
$$cr = c$$
$$as + dt = d$$
$$bs + et = e$$

Solving these five equations, we obtain $p = r = t = 1$ and $q = s = 0$

Thus,

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

is the identity rhotrix under row-column multiplication of dimension 3.

Result 2.3.2.2. Inverse of a rhotrix under heart-oriented multiplication is given by

$$Q = P^{-1} = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -b & \frac{ae-bd}{S} & -d \\ & a & \end{array} \right\rangle \quad (2.9)$$

Proof: Let the rhotrix $Q = \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle$ be the inverse of the rhotrix $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ under row-column multiplication.

Then, $P \circ Q = Q \circ P = I$

$$\Longrightarrow \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & p & \\ q & r & s \\ & t & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

$$\Longrightarrow \left\langle \begin{array}{ccc} & ap + dq & \\ bp + eq & cr & as + dt \\ & bs + et & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

$$\begin{aligned} \Longrightarrow ap + dq &= 1 \\ bp + eq &= 0 \\ cr &= 1 \\ as + dt &= 0 \\ bs + et &= 1 \end{aligned}$$

Solving these five equations, we obtain,

$$p = \frac{e}{ae - bd}, q = \frac{-b}{ae - bd}, r = \frac{1}{c}, s = \frac{-d}{ae - bd}, t = \frac{a}{ae - bd}$$

Thus, we have,

$$Q = P^{-1} = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -b & \frac{ae-bd}{S} & -d \\ & a & \end{array} \right\rangle$$

provided $c(ae - bd) \neq 0$.

2.4 The Rhotrix Exponent Rule

Theorem 2.4.1 *For a three-dimensional rhotrix* $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ *and for any integer values of m,*

$$R^m = (h(R))^{m-1} \left\langle \begin{array}{ccc} & ma & \\ mb & h(R) & md \\ & me & \end{array} \right\rangle \tag{2.10}$$

Here we use heart-oriented multiplication for finding the product of rhotrices.

Proof: This can be proved using principle of mathematical induction with induction on m.
For the case when $m = 1$,

$$R^1 = (h(R))^{1-1} \left\langle \begin{array}{ccc} & 1 \cdot a & \\ 1 \cdot b & h(R) & 1 \cdot d \\ & 1 \cdot e & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle = R$$

Now we suppose that the case when $m = k$ is true. That is,

$$R^k = (h(R))^{k-1} \left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle$$

Then for $m = k + 1$,

$$R^{k+1} = R^k \circ R$$

$$= (h(R))^{k-1} \left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

$$= (h(R))^{k-1} \left\langle \begin{array}{ccc} & kah(R) + ah(R) & \\ kbh(R) + bh(R) & h(R)h(R) & kdh(R) + dh(R) \\ & keh(R) + eh(R) & \end{array} \right\rangle$$

$$= (h(R))^{k-1} \left\langle \begin{array}{ccc} & (k+1)ah(R) & \\ (k+1)bh(R) & (h(R))^2 & (k+1)dh(R) \\ & (k+1)eh(R) & \end{array} \right\rangle$$

$$= (h(R))^{k-1+1} \left\langle \begin{array}{ccc} & (k+1)a & \\ (k+1)b & h(R) & (k+1)d \\ & (k+1)e & \end{array} \right\rangle$$

$$= (h(R))^{(k+1)-1} \left\langle \begin{array}{ccc} & (k+1)a & \\ (k+1)b & h(R) & (k+1)d \\ & (k+1)e & \end{array} \right\rangle$$

Thus, the case where $m = k + 1$ is true. Thus, by principle of mathematical induction, we can conclude that the theorem is true for all positive integers.

When m is a negative integer, write $m = -k$, so that k is a positive integer. Then,

$$R^m = R^{-k} = \frac{I}{R^k}$$

$$= \frac{I}{(h(R))^{k-1} \left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle}$$

$$= (h(R))^{-(k-1)} \left\langle \begin{array}{ccc} & ka & \\ kb & h(R) & kd \\ & ke & \end{array} \right\rangle^{-1}$$

$$= (h(R))^{-(k-1)} \frac{-1}{(h(R))^2} \left\langle \begin{array}{ccc} & ka & \\ kb & -h(R) & kd \\ & ke & \end{array} \right\rangle$$

$$= (h(R))^{-k-1} \left\langle \begin{array}{ccc} & -ka & \\ -kb & h(R) & -kd \\ & -ke & \end{array} \right\rangle$$

Thus the theorem is true for all negative integers.

Finally, for $m = 0$,

$$R^0 = 1 = I$$

and,

$$(h(R))^{0-1} \left\langle \begin{array}{ccc} & 0 \cdot a & \\ 0 \cdot b & h(R) & 0 \cdot d \\ & 0 \cdot e & \end{array} \right\rangle = \frac{1}{h(R)} \left\langle \begin{array}{ccc} & 0 & \\ 0 & h(R) & 0 \\ & 0 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle = I$$

Thus, for m=0, the theorem holds.

Thus we can conclude that the theorem is true for all $m \in \mathbb{Z}$.

Result 2.4.1. If R is a unit heart rhotrix, then we have

$$R^m = (h(R))^{m-1} \left\langle \begin{array}{ccc} & ma & \\ mb & h(R) & md \\ & me & \end{array} \right\rangle$$

$$= (1)^{m-1} \left\langle \begin{array}{ccc} & ma & \\ mb & 1 & md \\ & me & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ma & \\ mb & 1 & md \\ & me & \end{array} \right\rangle$$

2.4.1 Properties of the Rhotrix Exponent Rule

If m and n are integers, then for any real rhotrix R,

1. $R^m \circ R^n = R^{m+n}$
2. $\frac{R^m}{R^n} = R^{m-n}$
3. $(R^m)^n = R^{mn}$
4. $(kR)^m = k^m R^m$, where k is a scalar
5. $R^0 = I$, where I is the identity rhotrix
6. $R^{(-1)} = \frac{I}{R} = I \circ R^{-1}$, provided $h(R) \neq 0$
7. $R^m = 0$ (Zero rhotrix), provided $h(R) = 0$, and $m \geq 2$

2.5 Rhotrix Polynomials

A polynomial in an indeterminate x, over $\mathbb{R}$ is an expression of the form

$$f(x) = a_0 + a_1 x + ... + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

where $a_i \in \mathbb{R} \forall i$ and m is a positive integer. n is the degree of f if $a_n \neq 0$ and a_n is called the leading coefficient of f.

We call f a rhotrix polynomial if the indeterminate x is a rhotrix.

Example 2.5.1. The polynomial $f(R) = 2R^2 + R + 3$ where R is a three-dimensional rhotrix is a rhotrix polynomial.

When $R = \left\langle \begin{array}{ccc} & 3 & \\ 2 & 4 & 4 \\ & 1 & \end{array} \right\rangle$,

$$f(R) = 2 \cdot \left\langle \begin{array}{ccc} & 3 & \\ 2 & 4 & 4 \\ & 1 & \end{array} \right\rangle^2 + \left\langle \begin{array}{ccc} & 3 & \\ 2 & 4 & 4 \\ & 1 & \end{array} \right\rangle + 3 \cdot \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

$$= 2 \cdot \left\langle \begin{array}{ccc} & 3 & \\ 2 & 4 & 4 \\ & 1 & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccc} & 3 & \\ 2 & 4 & 4 \\ & 1 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & 3 & \\ 2 & 4 & 4 \\ & 1 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{array} \right\rangle$$

$$= 2 \cdot \left\langle \begin{array}{ccc} & 24 & \\ 16 & 16 & 32 \\ & 8 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & 3 & \\ 2 & 7 & 4 \\ & 1 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 48 & \\ 32 & 32 & 64 \\ & 16 & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & 3 & \\ 2 & 7 & 4 \\ & 1 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 51 & \\ 34 & 39 & 68 \\ & 17 & \end{array} \right\rangle$$

Remark 2.5.1. The rhotrix polynomial $f(R) = 0 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$ is called the zero rhotrix polynomial.

Chapter 3

APPLICATION OF RHOTRICES ON CRYPTOGRAPHY

In this section, we shall see two different techniques of encryption and decryption, each using heart-oriented multiplication and row-column multiplication. Both these processes follow a variation to Hill cipher, in the sense that rhotrices are used in stead of matrices as the cipher key.

3.1 Encryption using Heart-Oriented Multiplication

For encryption using rhotrices, we proceed using the following algorithm with the assumption that we have a specific message (plaintext) in hand to be encrypted (transformed into cipher text).

Step 1: **Identification of key rhotrix**

Fix a 3-dimensional rhotrix K, such that $h(K) = 1$.

We call this rhotrix, the key rhotrix of the process.

Step 2: **Finding the numerical equivalent of the plaintext**

Find the natural number corresponding to the alphabets in the message with the help of the following table:

A	B	C	D	E	F	G	H	I	J	K	L
1	2	3	4	5	6	7	8	9	10	11	12
M	N	O	P	Q	R	S	T	U	V	W	X
13	14	15	16	17	18	19	20	21	22	23	24

Table 3.1

Step 3: **Creating rhotrices corresponding to these numbers**

Divide the obtained sequence of numbers into groups of two in the same order of appearance. If the total number of numbers are odd, then put 0 after the last number to complete the pairs.
Use the first pair of numbers to form the first 3-dimensional rhotrix R_1, in which the two numbers form the first row. Set heart of the rhotrix to 1 to make the heart non-zero while maintaining the ease in calculations that follow. Fill the blank spaces in the last row with zero.
In a similar way, create second rhotrix R_2 using the second pair of numbers, create third rhotrix R_3 using the third pair of numbers, and so on.

Step 4: **Multiplying these rhotrices with the key rhotrix**

Multiply the rhotrix $R_i, i = 1, 2, 3, \ldots$ obtained from Step 3 with the key rhotrix K using heart-oriented multiplication to create the rhotrix R_i'.

Step 5: **Retract the numbers from R_i'**

To retract the numbers from R_i', consider the first row and take those numbers as pairs.
Form a sequence using these numbers. If any of these numbers are greater than 26 or less than 0, take the residue modulo 26 of that number and replace them in the respective position in the sequence (if the residue is 0, replace the number in the sequence by 26).

Step 6: **Finding the encrypted message**

Find the alphabetical equivalent of the numbers in the above sequence using the table mentioned in Step 2.

These letters make up the ciphertext.

Example 3.1.1. Encrypting the word **INDIA** using the above method:

Fix the key rhotrix $K = \left\langle \begin{array}{ccc} & 2 & \\ 4 & 1 & 1 \\ & 3 & \end{array} \right\rangle$.

The numerical equivalent of the word INDIA is 9, 14, 4, 9, 1.

We shall add 0 after the last number since we have a total of 5 (which is odd) numbers.

Then the above sequence becomes 9, 14, 4, 9, 1, 0.

Divide these numbers into groups of two. Thus we have three pairs $(9, 14), (4, 9), (1, 0)$.

Create three rhotrices R_1, R_2 and R_3 corresponding to these pairs.

Thus, we have,

$$R_1 = \left\langle \begin{array}{ccc} & 9 & \\ 0 & 1 & 14 \\ & 0 & \end{array} \right\rangle$$

$$R_2 = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 9 \\ & 0 & \end{array} \right\rangle$$

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

Now we form the rhotrices R_1',R_2' and R_3' by multiplying R_1,R_2 and R_3 by K.

Thus, we have,

$$R_1{}' = \left\langle \begin{array}{ccc} & 9 & \\ 0 & 1 & 14 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 4 & 1 & 1 \\ & 3 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 11 & \\ 4 & 1 & 15 \\ & 3 & \end{array} \right\rangle$$

$$R_2{}' = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 9 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 4 & 1 & 1 \\ & 3 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 6 & \\ 4 & 1 & 10 \\ & 3 & \end{array} \right\rangle$$

$$R_3{}' = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 4 & 1 & 1 \\ & 3 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 3 & \\ 4 & 1 & 1 \\ & 3 & \end{array} \right\rangle$$

Therefore, we have the sequence corresponding to $R_1{}'$,$R_2{}'$ and $R_3{}'$ as 11, 15, 6, 10, 3, 1.
The alphabetical equivalent to these numbers is **KOFJCA**.
This is our encrypted message.

3.2 Decryption using Heart-Oriented Multiplication

For decryption using rhotrices, we proceed using the following algorithm with the assumption that we have an encrypted text (cipher text) in hand to be decrypted (transformed into plaintext) and the key rhotrix K with which the plaintext was initially encrypted.

Step 1: **Finding the inverse of key rhotrix**

Find the inverse K^{-1} of the key rhotrix K.

Step 2: **Finding the numerical equivalent of the cipher text**

Find the natural number corresponding to the alphabets in the encrypted message with the help of Table 3.1 mentioned in Step 2 of the algorithm for encryption using Heart-Oriented multiplication

Step 3: **Creating rhotrices corresponding to these numbers**

Divide the obtained sequence of numbers into groups of two in the same order of appearance. The encryption algorithm ensures that there will always be even number of numbers so that each number will have a pair. Use the first pair of numbers to form the first 3-dimensional rhotrix R_1', in which the two numbers form the first row. Set heart of the rhotrix to 1 to make the heart non-zero while maintaining the ease in calculations that follow. Fill the blank spaces in the last row with zero.
In a similar way, create second rhotrix R_2' using the second pair of numbers, create third rhotrix R_3' using the third pair of numbers, and so on.

Step 4: **Multiplying these rhotrices with the inverse of key rhotrix**

Multiply the rhotrix $R_i', i = 1, 2, 3, \ldots$ obtained from Step 3 with the inverse K^{-1} of the key rhotrix K using heart-oriented multiplication to create the rhotrix R_i.

Step 5: **Retract the numbers from R_i'**

To retract the numbers from R_i, consider the first row of R_i and take those numbers as pairs.
Form a sequence using these numbers. If any of these numbers are greater than 26 or less than 0, take the residue modulo 26 of that number and replace them in the respective position in the sequence (if the residue is 0, replace the number in the sequence by 26).
Also, if the last number in the sequence is 0, remove it from the sequence.

Step 6: **Finding the encrypted message**

Find the alphabetical equivalent of the numbers in the above sequence using Table 3.1 mentioned in Step 2 of the algorithm for encryption using Heart-Oriented multiplication.
These letters gives us the plaintext.

Example 3.2.1. Decrypting the word **KOFJCA** which was encrypted using the key rhotrix $K = \left\langle \begin{array}{ccc} & 2 & \\ 4 & 1 & 1 \\ & 3 & \end{array} \right\rangle$ by following the above algorithm:

The inverse of the key rhotrix is $K^{-1} = \left\langle \begin{array}{ccc} & -2 & \\ -4 & 1 & -1 \\ & -3 & \end{array} \right\rangle$.

The numeric equivalent of the word KOFJCA is 11, 15, 6, 10, 3, 1.

Divide these numbers into groups of two. Thus we have three pairs (11, 15), (6, 10), (3, 1).

Create three rhotrices R_1',R_2' and R_3' corresponding to these pairs.

Thus, we have,

$$R_1' = \left\langle \begin{array}{ccc} & 11 & \\ 0 & 1 & 15 \\ & 0 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 6 & \\ 0 & 1 & 10 \\ & 0 & \end{array} \right\rangle$$

$$R_3' = \left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle$$

Now we form the rhotrices R_1,R_2 and R_3 by multiplying R_1',R_2' and R_3' by K^{-1}.

Thus, we have,

$$R_1' = \left\langle \begin{array}{ccc} & 11 & \\ 0 & 1 & 15 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -2 & \\ -4 & 1 & -1 \\ & -3 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 9 & \\ -4 & 1 & 14 \\ & -3 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 6 & \\ 0 & 1 & 10 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -2 & \\ -4 & 1 & -1 \\ & -3 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 4 & \\ -4 & 1 & 9 \\ & -3 & \end{array} \right\rangle$$

$$R_3' = \left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -2 & \\ -4 & 1 & -1 \\ & -3 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ -4 & 1 & 0 \\ & -3 & \end{array} \right\rangle$$

Therefore, we have the sequence corresponding to R_1,R_2 and R_3 as 9, 14, 4, 9, 1,0.
Removing the last number 0 from the sequence, we get the sequence 9, 14, 4, 9, 1.
The alphabetical equivalent to these numbers is **INDIA**.
This is our plaintext.

Example 3.2.1. Encrypting and decrypting the word **MATHEMATICS** Fix the key rhotrix $K = \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle$.

The numerical equivalent of the word MATHEMATICS is 13, 1, 20, 8, 5, 13, 1, 20, 9, 3, 19.
We shall add 0 after the last number since we have a total of 11 (which is odd) numbers.
Then the above sequence becomes 13, 1, 20, 8, 5, 13, 1, 20, 9, 3, 19, 0.
Divide these numbers into groups of two. Thus we have six pairs $(13, 1), (20, 8), (5, 13), (1, 20), (9,$
Create six rhotrices $R_1, R_2, ..., R_6$ corresponding to these pairs. Thus, we have,

$$R_1 = \left\langle \begin{array}{ccc} & 13 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle$$

$$R_2 = \left\langle \begin{array}{ccc} & 20 & \\ 0 & 1 & 8 \\ & 0 & \end{array} \right\rangle$$

$$R_3 = \left\langle \begin{array}{ccc} & 5 & \\ 0 & 1 & 13 \\ & 0 & \end{array} \right\rangle$$

$$R_4 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 20 \\ & 0 & \end{array} \right\rangle$$

$$R_5 = \left\langle \begin{array}{ccc} & 9 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle$$

$$R_6 = \left\langle \begin{array}{ccc} & 19 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

Now we form the rhotrices $R_1', R_2', ..., R_6'$ by multiplying $R_1, R_2, ..., R_6$ by K.

Thus, we have,

$$R_1' = \left\langle \begin{array}{ccc} & 13 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 23 & \\ 8 & 1 & 17 \\ & 17 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 20 & \\ 0 & 1 & 8 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 30 & \\ 8 & 1 & 24 \\ & 17 & \end{array} \right\rangle$$

$$R_3' = \left\langle \begin{array}{ccc} & 5 & \\ 0 & 1 & 13 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 15 & \\ 8 & 1 & 29 \\ & 17 & \end{array} \right\rangle$$

$$R_4' = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 20 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 11 & \\ 8 & 1 & 36 \\ & 17 & \end{array} \right\rangle$$

$$R_5{}' = \left\langle \begin{array}{ccc} & 9 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 19 & \\ 8 & 1 & 19 \\ & 17 & \end{array} \right\rangle$$

$$R_6{}' = \left\langle \begin{array}{ccc} & 19 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 29 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle$$

Therefore, we have the sequence corresponding to $R_1{}', R_2{}', ...R_6{}'$ as 23, 17, 30, 24, 15, 29, 11, 36, 19, 19, 29, 16.

The sequence obtained after taking residue modulo 26 is 23, 17, 4, 24, 15, 3, 11, 10, 19, 19; 3, 16.

The alphabetical equivalent to these numbers is **WQDXOCKJSSCP**.

This is our encrypted message.

Now, we shall decrypt the message **WQDXOCKJSSCP** which was encrypted using the key rhotrix $K = \left\langle \begin{array}{ccc} & 10 & \\ 8 & 1 & 16 \\ & 17 & \end{array} \right\rangle$.

The inverse of the key rhotrix is $K^{-1} = \left\langle \begin{array}{ccc} & -10 & \\ -8 & 1 & -16 \\ & -17 & \end{array} \right\rangle$.

The numeric equivalent of the word WQDXOCKJSSCP is 23, 17, 4, 24, 15, 3, 11, 10, 19, 19, 3, 16.

Divide these numbers into groups of two. Thus we have six pairs (23, 17), (4, 24), (15, 3), (11, 10), (19, 19), (3, 16).

Create three rhotrices $R_1{}', R_2{}', ..., R_6{}'$ corresponding to these pairs.

Thus, we have,

$$R_1{}' = \left\langle \begin{array}{ccc} & 23 & \\ 0 & 1 & 17 \\ & 0 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 24 \\ & 0 & \end{array} \right\rangle$$

$$R_3' = \left\langle \begin{array}{ccc} & 15 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle$$

$$R_4' = \left\langle \begin{array}{ccc} & 11 & \\ 0 & 1 & 10 \\ & 0 & \end{array} \right\rangle$$

$$R_5' = \left\langle \begin{array}{ccc} & 19 & \\ 0 & 1 & 19 \\ & 0 & \end{array} \right\rangle$$

$$R_6' = \left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 16 \\ & 0 & \end{array} \right\rangle$$

Now we form the rhotrices $R_1, R_2, ..., R_6$ by multiplying $R_1', R_2', ..., R_6'$ by K^{-1}. Thus, we have,

$$R_1' = \left\langle \begin{array}{ccc} & 23 & \\ 0 & 1 & 17 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -10 & \\ -8 & 1 & -16 \\ & -17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 13 & \\ -8 & 1 & 1 \\ & -17 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 24 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -10 & \\ -8 & 1 & -16 \\ & -17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & -6 & \\ -8 & 1 & 8 \\ & -17 & \end{array} \right\rangle$$

$$R_3' = \left\langle \begin{array}{ccc} & 15 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -10 & \\ -8 & 1 & -16 \\ & -17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 5 & \\ -8 & 1 & -13 \\ & -17 & \end{array} \right\rangle$$

$$R_4{}' = \left\langle \begin{array}{ccc} & 11 & \\ 0 & 1 & 10 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -10 & \\ -8 & 1 & -16 \\ & -17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ -8 & 1 & -6 \\ & -17 & \end{array} \right\rangle$$

$$R_5{}' = \left\langle \begin{array}{ccc} & 19 & \\ 0 & 1 & 19 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -10 & \\ -8 & 1 & -16 \\ & -17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 9 & \\ -8 & 1 & 3 \\ & -17 & \end{array} \right\rangle$$

$$R_6{}' = \left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 16 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & -10 & \\ -8 & 1 & -16 \\ & -17 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & -7 & \\ -8 & 1 & 0 \\ & -17 & \end{array} \right\rangle$$

Therefore, we have the sequence corresponding to $R_1, R_2, ..., R_6$ as 13, 1, -6, 8, 5, -13, 1, -6, 9, 3, -7, 0.

The sequence obtained after taking residue modulo 26 is 13, 1, 20, 8, 5, 13, 1, 20, 9, 3, 19, 0.

Removing the last number 0 from the sequence, we get the sequence 13, 1, 20, 8, 5, 13, 1, 20, 9, 3, 19.

The alphabetical equivalent to these numbers is **MATHEMATICS**.

This is our plaintext.

3.3 Encryption using Row-Column Multiplication

For encryption using rhotrices, we proceed using the following algorithm with the assumption that we have a specific message (plaintext) in hand to be encrypted (transformed into cipher text).

Step 1: **Identification of key rhotrix**

Fix a 3-dimensional rhotrix $K = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$, such that $c(ae - bd) \neq 0$.

We call this rhotrix, the key rhotrix of the process.

Step 2: **Finding the numerical equivalent of the plaintext**

Find the natural number corresponding to the alphabets in the message with the help of Table 3.1 mentioned in Step 2 of Section 3.1

Step 3: **Creating rhotrices corresponding to these numbers**

Divide the obtained sequence of numbers into groups of two in the same order of appearance. If the total number of numbers are odd, then put 0 after the last number to complete the pairs.
Use the first pair of numbers to form the first 3-dimensional rhotrix R_1, in which the two numbers form the first column and fill the remaining three spaces with zeros.
In a similar way, create second rhotrix R_2 using the second pair of numbers, create third rhotrix R_3 using the third pair of numbers, and so on.

Step 4: **Multiplying these rhotrices with the key rhotrix**

Multiply the rhotrix R_i,$i = 1, 2, 3, \ldots$ obtained from Step 3 to the right of the key rhotrix K using row-column multiplication to create the rhotrix R_i'.

Step 5: **Retract the numbers from R_i'**

To retract the numbers from R_i', consider the first column and take those numbers as pairs.
Form a sequence using these numbers. If any of these numbers are greater than 26 or less than 0, take the residue modulo 26 of that number and replace them in the respective position in the sequence (if the residue is 0, replace the number in the sequence by 26).

Step 6: **Finding the encrypted message**

Find the alphabetical equivalent of the numbers in the above sequence using the table mentioned in Step 2.
These letters make up the ciphertext.

3.4 Decryption using Row-Column Multiplication

For decryption using rhotrices, we proceed using the following algorithm with the assumption that we have an encrypted text (cipher text) in hand to be decrypted (transformed into plaintext) and the key rhotrix K with which the plaintext was initially encrypted.

Step 1: **Finding the inverse of key rhotrix**

Find the inverse K^{-1} of the key rhotrix K.

Step 2: **Finding the numerical equivalent of the cipher text**

Find the natural number corresponding to the alphabets in the encrypted message with the help of Table 3.1 mentioned in Step 2 of Section 3.1

Step 3: **Creating rhotrices corresponding to these numbers**

Divide the obtained sequence of numbers into groups of two in the same order of appearance. The encryption algorithm in Section 3.3 ensures that there will always be even number of numbers so that each number will have a pair.
Use the first pair of numbers to form the first 3-dimensional rhotrix R_1', in which the two numbers form the first column. Fill the blank spaces in the remaining two columns with zero.
In a similar way, create second rhotrix R_2' using the second pair of numbers, create third rhotrix R_3' using the third pair of numbers, and so on.

Step 4: **Multiplying these rhotrices with the inverse of key rhotrix**

Multiply the rhotrix $R_i', i = 1, 2, 3, \ldots$ obtained from Step 3 to the right of the inverse K^{-1} of the key rhotrix K using row-column multiplication to create the rhotrix R_i.

Step 5: **Retract the numbers from** R_i'

To retract the numbers from R_i, consider the first column of R_i and take those numbers as pairs.

Form a sequence using these numbers. If any of these numbers are greater than 26 or less than 0, take the residue modulo 26 of that number and replace them in the respective position in the sequence (if the residue is 0, replace the number in the sequence by 26).

Also, if the last number in the sequence is 0, remove it from the sequence.

Step 6: **Finding the encrypted message**

Find the alphabetical equivalent of the numbers in the above sequence using Table 3.1 mentioned in Step 2 of Section 3.1.

These letters gives us the plaintext.

Example 3.4.2. Encrypting and decrypting the word **SAVE** Fix the key rhotrix

$$K = \left\langle \begin{array}{ccc} & 2 & \\ 3 & 1 & 1 \\ & 2 & \end{array} \right\rangle.$$

The numerical equivalent of the word SAVE is 19, 1, 22, 5.

Divide these numbers into groups of two. Thus we have two pairs $(19, 1), (22, 5)$.

Create two rhotrices R_1 and R_2 corresponding to these pairs. Thus, we have,

$$R_1 = \left\langle \begin{array}{ccc} & 19 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

$$R_2 = \left\langle \begin{array}{ccc} & 22 & \\ 5 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

Now we form the rhotrices R_1' and R_2' by multiplying R_1 and R_2 by K.

Thus, we have,

$$R_1' = \left\langle \begin{array}{ccc} & 2 & \\ 3 & 1 & 1 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 19 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 39 & \\ 59 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 2 & \\ 3 & 1 & 1 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 22 & \\ 5 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 49 & \\ 76 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

Therefore, we have the sequence corresponding to R_1' and R_2' as 39, 59, 49, 76.
The sequence obtained after taking residue modulo 26 is 13, 7, 23, 24.
The alphabetical equivalent to these numbers is **MGWX**.
This is our encrypted message.

Now, we shall decrypt the message **MGWX** which was encrypted using the key rhotrix $K = \left\langle \begin{array}{ccc} & 2 & \\ 3 & 1 & 1 \\ & 2 & \end{array} \right\rangle$.

The inverse of the key rhotrix is $K^{-1} = \left\langle \begin{array}{ccc} & 2 & \\ -3 & 1 & -1 \\ & 2 & \end{array} \right\rangle$.
The numeric equivalent of the word MGWX is 13, 7, 23, 24.
Divide these numbers into groups of two. Thus we have two pairs (13, 7), (23, 24).
Create three rhotrices R_1' and R_2' corresponding to these pairs.
Thus, we have,

$$R_1' = \left\langle \begin{array}{ccc} & 13 & \\ 7 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 13 & \\ 7 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

Now we form the rhotrices R_1 and R_2 by multiplying R_1' and R_2' by K^{-1}. Thus, we have,

$$R_1' = \left\langle \begin{array}{ccc} & 2 & \\ -3 & 1 & -1 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 13 & \\ 7 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 19 & \\ -25 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

$$R_2' = \left\langle \begin{array}{ccc} & 2 & \\ -3 & 1 & -1 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 13 & \\ 7 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 22 & \\ -21 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

Therefore, we have the sequence corresponding to R_1 and R_2 as 19, -25, 22, -21.

The sequence obtained after taking residue modulo 26 is 19, 1, 22, 5.

The alphabetical equivalent to these numbers is **SAVE**.

This is our plaintext.

Conclusion

Rhotrix theory is a relatively new area of study in the field of mathematics. Rhotrices, a unique category of matrices known for their rhomboidal structure, has been an area of interest of researchers, especially in the field of Linear algebra since its inception. In this project, we take a closer look at the fundamental aspects of rhotrices, elaborating on their structure, operations on them, and an application in the field of cryptography.

This project begins with the examination of the structure of a rhotrix. Key terms and meanings that are required to establish a solid foundation for a deeper understanding of rhotrices were introduced. Moving forward, the project delves into the operations performed on rhotrices. This includes various linear operations like the addition of rhotrices, multiplication of a rhotrix by a scalar, and the different methods of multiplying two rhotrices, and also reflecting on their algebraic properties. Additionally, the Rhotrix Exponent Rule and the concept of rhotrix polynomials will also be dealt with.

As we conclude, our focus shifts to the practical application of rhotrices in cryptography. Here, these unique properties of rhotrices play a crucial role in introducing an innovative technique of encrypting and decrypting texual data. The concept of heart-oriented multiplication becomes the cornerstone of this process, ensuring security. In essence, the presentation aims to provide an introduction to the foundational concepts of rhotrices and an insight on how to make use of the theory of rhotrices to enable rhotrices to be the crux of a cryptographic strategy.

Thus, through this project, we can see that Rhotrix theory provide a fertile ground for mathematical research. Also, it is important to make sure that the potential of rhotrices to open up a new approach of securing messages shall not be overlooked

REFERENCES

[1] A. O. Ajibade, "The Concept of Rhotrix in Mathematical Enrichment." *International Journal of Mathematical Education in Science and Technology*, Vol. 34, No. 2, 2003, pp.175-179.

[2] B. Sani, "An Alternative Method for Multiplication of Rhotrices."International Journal of Mathematical Education in Science and Technology, Vol. 35, No. 5, 2004, pp. 777-781.

[3] B. Sani, "The Row-Column Multiplication of Higher Dimensional Rhotrices. "International Journal of Mathematical Education in Science and Technology, Vol. 38, No. 5,2007, pp. 657-662

[4] Ezugwu, Absalom Abdullahi, Mohammed B, Sani Junaidu, Sahalu. (2011). APPLICATION OF STRASSEN'S ALGORITHM IN RHOTRIX ROW-COLUMN MULTIPLICATION.

[5] J. B. Fraleigh,"A First Course in Abstract Algebra ",7th Edition, Addison-Wesley, Boston, 1982.

[6] Kenneth Hoffman and Ray Kunze. "Linear Algebra", Second edition. Prentice Hall India Learning Private Limited (1978)

[7] Mohammed, A. "A note on rhotrix exponent rule and its applications to some special series and polynomial equations defined over rhotrices ". Notes on Number Theory and Discrete Mathematics, 13(1), 2007, pp. 1-15.

[8] Mohammed, A. and Balarabe, M. (2014) First Review of Articles on Rhotrix Theory Since Its Inception. Advances in Linear Algebra Matrix Theory, 4, 216-224.

[9] S. M. Tudunkaya. Rhotrix Polynomials and Polynomial Rhotrices, Pure and AppliedMathematics Journal. Vol. 2, No. 1, 2013, pp. 38-41.

[10] https://www.geeksforgeeks.org/cryptography-and-its-types/

[11] https://www.geeksforgeeks.org/hill-cipher/