Project Report

On

STURM-LIOUVILLE PROBLEMS AND ITS APPLICATIONS

Submitted

in partial fulfilment of the requirements for the degree of

BACHELOR OF SCIENCE

in

MATHEMATICS

by

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CERTIFICATE

This is to certify that the dissertation entitled, **STURM-LIOUVILLE PROBLEMS AND ITS APPLICATIONS** is a bonafide record of the work done by Ms. **SREELAKSHMI MURALI** under my guidance as partial fulfillment of the award of the degree of **Bachelor of Science in Mathematics** at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

Date:25/02/2023
Place: Ernakulam

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DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Donna Pinheiro, Assistant Professor, Department of Mathematics, St. Teresa's College(Autonomous), Ernakulam and has not been included in any other project submitted previously for the award of any degree.

Ernakulam. **SREELAKSHMI MURALI**

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I must mention several individuals who encouraged me to carry this work.Their continuous invaluable knowledgably guidance throughout the course of this study helped me to complete the work up to this stage.

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Ernakulam. SREELAKSHMI MURALI

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Chapter 1

INTRODUCTION

In mathematics and its applications, STURM – LIOUVILLE THEORY play an important role in the existence of the Eigen values and Eigen functions. A boundary value problem consists of solving an ordinary differential equation in a given interval, subject to certain boundary conditions. This boundary value problem becomes a Sturm – Liouville problem if it is a second order differential equation of a specific form, together with two boundary conditions. The purpose of our project is to study in detail the Sturm – Liouville problems and related methods for finding the Eigen values and Eigen functions.
A Sturm – Liouville problem is a second order differential equation of the form:
$(p(x)y^{'})^{'} - q(x) + \lambda r(x)y = 0, a < x < b$

Together with the boundary conditions:
$a_1y(0) + a_2y^{'}(0) = 0$
$b_1y(1) + b_2y^{'}(1) = 0$

Here p, q and r are specific functions and λ is a parameter.Here a and b are not both zero. The solution of Sturm – Liouville problem consists of finding the values for λ for which the corresponding solution must be nontrivial and satisfying the boundary conditions. When such value of λ exists, they are called Eigen values and the corresponding solutions are called Eigen functions. A Sturm - Liouville problem is called regular if the functions q(x) and r(x) are continuous on [a,b]

and also p(x) is differentiable on the same interval. Many Ordinary Differential Equations that occur during separation of variables can be put in the Sturm-Liouville form. For example, Bessel equation, Legendre's Differential equation, Chebyshev's Differential equation etc. can be reduced to this form.

1.1 HISTORY

Joseph Liouville was a French mathematician and engineer who is known for his discovery of transcendental numbers which are the numbers that are not the roots of the algebraic equations having rational coefficients. Jacques Charles François Sturm was a French Mathematician who is known for Sturm Separation theorem, Sturm Series, Sturm's Theorem, Sturm – Liouville Theory, Sturm-Picone Comparison Theorem and Speed of Sound.

In a series of papers dating back to 1836-1837, STURM and LIOUVILLE created an entirely new subject of mathematical analysis. The theory developed was later known as Sturm – Liouville theory . The questions studied by STURM and LIOUVILLE can roughly be divided into three groups :

1. Properties of the eigen values

2. Qualitative behavior of the eigen functions

3. Expansion of arbitrary functions in an infinite series of eigen functions .

Of these , Sturm examined the properties of the eigen values and qualitative behavior of the eigen functions and Liouville studied the expansion of arbitrary functions in an infinite series of eigen functions giving further results for Sturm's examinations . Until 1820, the only question that occupied differential equation theory was to find a solution in the form of an analytic expression , given a differential equation. Sturm could not find such an expression , which Liouville found by successive approximation and was not suitable for studying the above properties . Instead information about the properties of the solution was obtained from the equation itself . It shows proof of a new concept in differential

equation theory, and is characterized by a broader type of question to investigate a particular property of a solution for a given differential equation. The Sturm-Liouville theory provided the first theorem for the eigenvalue problem and occupies a central place in the prehistory of functional analysis. But the Sturm-Liouville theory is important not only as a herald of future ideas. It is important even today for the technical treatment of many specific problems in pure and applied mathematics, and not merely in conceptual sense. The Strum's theorem is an important contribution to the theory of equations. The Sturm-Liouville equation helps to explain many important physical processes and mechanical systems in classical and quantum physics. In many cases, the Sturm-Liouville problem describes the oscillations in a physical system.

1.2 SIGNIFICANCE OF THE STUDY

The Sturm-Liouville Theory is used to solve many day to day problems associated with the partial differentiation and second order linear differential equation. For example, in quantum mechanics, the one-dimensional time-independent Schrodinger equation is a Sturm–Liouville problem:

$$\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \tag{1.1}$$

$\hbar$= The planks constant as $\frac{h}{2\pi}$ which is a constant of action in tthe dynamic geometrical process that we see and feel as the passage of the time

∇^2 = this quantity describes how the wave function ψ changes from one moment to another

ψ = wave function

m= Mass of the particle

V= Force acting on the particle

E=Energy

This theory also helps us in reducing the complexity of many partial differential equations. The Heat Equation can also be converted to a

Sturm -Liouville problem : Consider the Heat equation,

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K_0\frac{\partial u}{\partial x} + \alpha u) \tag{1.2}$$

Where

c=Specific Heat Capacity

ρ**=Density**

α**=Material's Thermal Diffusity**

K_0**=Material's Conductivity**

which is a homogenous equation where we can apply separation of variables,

$u(x,t) = \phi(x)h(t)$ **to the Partial Differential Equation and rearrange to**

$\frac{d}{dx}(K_0\frac{d\phi}{dx}) + \alpha\phi + \lambda c\rho\phi = 0$

This is a Sturm-Liouville Problem, if there are homogeneous Boundary Conditions. Also Sturm-Liouville problem has many applications in the physical world such as Sound wave analysis ,Image processing ,Quantum mechanics, Electrical networks etc

Chapter 2

NUMERICAL METHODS FOR SOLVING BOUNDARY VALUE PROBLEMS

There are many mathematical methods for solving Boundary value problems such as finite difference method, variational estimation method, shooting method etc. Finite difference method requires large amount of memory for matrices and need lot of arithmetic calculations. If finite difference method is used to solve an eigen value problem, it decreases in accuracy when the eigen values are large.
In the next section we are discussing about solving boundary value problems by shooting method and finite difference method.

2.1 SHOOTING METHOD

Shooting Method is a way of approximating a solution of a boundary value problem

$$y'' = f(x,y,y') \text{ ,where } y(a) = \alpha \text{ ;} y(b) = \beta$$

This method basically uses the initial value problem methods for solving the equations .
The first step is the replacement of the boundary-value problem by an initial-value problem

y" = f(x,y,y') , where y(a) = α; y'(a) =m_1

The m_1 in the above equation is just a guess for the unknown slope of the solution curve at the known point (a, y(a)). Then we apply one of the step-by-step numerical techniques to the above second-order differential equation to find an approximation β_1 for the value of y(b). We can use any of the methods for solving initial value problem such as Euler's Method , Runge Kutta Second Order Method , Runge Kutta Forth Order Method etc .

If β_1 agrees with the given value y(b) = β to some preassigned tolerance, we stop; otherwise, the calculations are repeated, starting with a different guess y'(a) = m_2 to obtain a second approximation β_2 for y(b). This method can be continued in a trial-and-error manner, or the subsequent slopes m_3 ,m_4 . . . can be adjusted in some systematic way . The procedure is analogous to shooting (here the aim is the choice of the initial slope) at a target until the bull's-eye y(b) is hit.

The working of shooting method :

a) The result with the first guess

b) The result with the second guess

c) The expected result

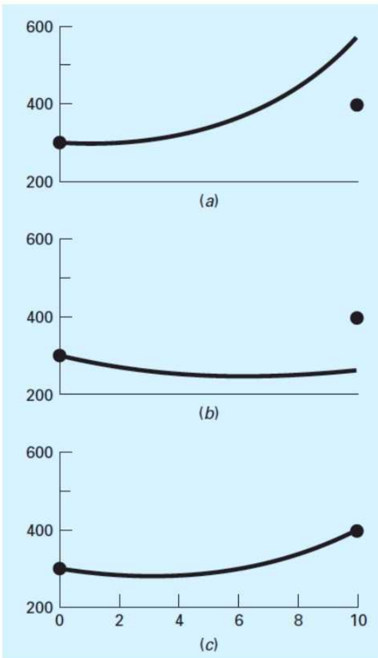


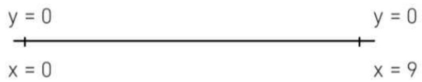
Figure 2.1: Working of shooting method

EXAMPLE:

Consider the Boundary value problem
y"-2y = 8x(9-x) , y(0) = 0 ; y(9) = 0
Use the shooting method to approximate the solution of this problem.

SOLUTION:

Given x = 0 and x = 9 are the two boundary conditions where the value of y is 0 and 0 respectively .



We can use any initial value problem method . Here we use Euler's Method and let the step size be h = 3 .

For converting the equation to a initial value equation ,
Let us assume that y'(0) = 4 ; y(0)=0
we are going to check whether the value of y at x = 9 is 0 . Most probabily we do not get 0 .
So we will change this initial value of y'(0) so that the value would be approximately equal to 0 .
We can write y" = f(x,y,y') ie; y" = 2y + 8x (9 – x), y(0) = 0;y'(0) = 4
We are introducing a new variable "z".
ie; y'= z
$y' = f_1(x, y, z)$, **y(0) = 0 ———–(1)**

and

y" = 2y + 8x (9 − x)

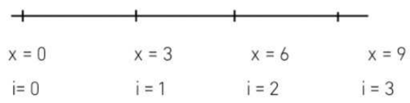
= $f_2(x, y, z)$, **z(0)= y'(0) = 4 ————-(2)**

By Euler's Method ,

Equation (1) implies , $Y_{i+1} = Y_i + f_1(x_i, y_i, z_i)h$

Equation (2) implies , $Z_{i+1} = Z_i + f_2(x_i, y_i, z_i)h$

When i = 0



$Y_1 = Y_0 + f_1(x_0, y_0, z_0)h$

$Z_1 = Z_0 + f_1(x_0, y_0, z_0)h$

We have , x_0 **= 0 ,** $y_0 = 0$ **,** $z_0 = 4$ **, h = 3**

Therefore $Y_1 = 0 + f_1(0, 0, 4) \cdot 3$

= 0 + (4 · 3)

= 12

and

$Z_1 = 4 + f_2(0, 0, 4) \cdot 3$

= 4 + (2y + 8x(9-x))

$= 4 + (2 \cdot 0 + 8 \cdot (9 - 0))$

= 4

$Y(x_1) \approx y_1 = 12$

$Z(x_1) \approx z_1 = 4$ **where** $x_1 = 3$

When i = 1

$Y_2 = y_1 + f_1(x_1, y_1, z_1)h$

$Z_2 = Z_1 + f_2(x_1, y_1, z_0)h$

We have , $x_1 = 3, y_1 = 12, z_1 = 4, h = 3$

Therefore $y_2 = 12 + f_1(3, 12, 4) \cdot 3$

= 12 +(4 ·3)

= 24

and

$z_1 = 4 + f_2(3, 12, 4) \cdot 3$

$= 4 + (2 \cdot 12 + 8 \cdot 3 \cdot (9 - 3)) \cdot 3$

$= 4 + (168 \cdot 3)$

$\mathbf{= 508}$

$y(x_1) \approx 24$

$z(x_2) \approx 508$ **where**$x_2 = 6$

When i=2

$Y_3 = Y_2 + f_1(x_2, y_2, z_2)h$

$z_3 = z_2 + f_2(x_2, y_2, z_2)h$

we have$x_2 = x_1 + h = 3 + 3 = 6$

$y_2 = 24$

$z_2 = 508$

therefore $y_3 = 24 + f_1(6, 24, 508) \cdot 3$

$= 24 + (508 \cdot 3)$

$\mathbf{= 1548}$

$y(x_3) \approx y_3 = 1548$

$y'(0) = 4 \to y(9) \approx 1548$

$y'(0) =? \to y(9) \approx ?$

So , $y_3 = 1548 \approx y(9)$

Choose , y'(0) = -24

Proceeding with Euler's Method

i	Xi	yi	Zi
0	0	0	-24
1	3	-72	-24
2	6	-144	-24
3	9	-216	

$y_3 = -216 \approx y(9)$

So ,

so we have to find for what value of y'(0) we get $y - 3 = 0$

Y'(0)	y3 ≈ y(9)
4	1548
-24	-216
p	q

let p_0**=4 ,** p_1**=-24 ,** q_0**=1548 ,** q_1**=-216 , and y'(0) = p and** $y_3 = q$

Then ,

$p = p_0 + \frac{(p_1 - p_0)}{(q_1 - q_0)}(q - q_0)$

$= 4 + \frac{(-24-4)}{(-216-1548)})(q - 1548)$ **where q=0**

= -20.57

Choose , y'(0) = 20.57

Proceeding with Euler's Method ,

$y_3 = 0.09 \approx y(9)$

i	x	y	z
0	0	0	-20.57
1	3	-61.7	-20.57
2	6	-123.42	41.17
3	9	0.09	

so we can see that 0.09 is getting close to the value of y(9) .

2.2 FINITE DIFFERENCE METHOD

consider a linear second-order boundary value problem given by:

y" + p(x)y' + q(x)y = f(x) ;y(a) = a_0**, y(b) =**b_0

represents a regular partition of the interval [a, b], that is x_i **can be written as,** x_i **= a + ih, where i = 0, 1, 2, . . . , n and h =**$\frac{(b-a)}{n}$**.**

The points $x_1, x_2, \ldots, x_{n-1}$ **given by**

x_1 **= a + h,**

x_2 **= a + 2h,**

.

.

.

x_{n-1} **= a + (n - 1)h**

are called interior mesh points of the interval [a, b].

If we take, $y_i = y(x_i), p_i = p(x_i), q_i = q(x_i), andf_i = f(x - i)$ **and if y" and y' in the above boundary value problem are replaced by the central difference approximations**

$y'(x) \approx \frac{[y(x+h) - y(x-h)]}{2h}$

$y''(x) \approx \frac{[y(x+h) - 2y(x) + y(x-h)]}{h^2}$

we get ,

$(1 + \frac{hP_i}{2})y_{i+1} + (-2 + h2q_i)y_i + (\frac{hP_i}{2})y_{i-1} = h^2 f - i$

This equation is known as finite difference equation. It is an approxima-

tion to the differential equation. It helps to approximate the solution of the given differential equation at the points (interior mesh points) x_1, x_2 , ...,x_{n-1} **of the interval [a, b]. By letting i take on the values 1, 2, . . . , n - 1 in the final approximation, we obtain n - 1 equations in the n – 1 unknowns** $y_1, y_2, ..., y_{n-1}$.

On solving we get

$(2 - hp_i)y_{i+1} - (4 + 2h^2q_i)y_i + (2 + hp_i)yi - 1 = 2h^2r_i$

$(2 + hp_i)yi - 1 - (4 + 2h^2q_i)y_i + (2 - hp_i)y_{i+1} = 2h^2r_i$

This is a tridiagonal system. Where y_0 **and**y_n, **since these are the prescribed boundary conditions** $y_0 = y(x_0) = y(a) = a_0$**and** $y_n = y(x_n) = y(b) = b_0$.

Using the boundary conditions and simplifying ,we get

i=1,$-(4 + 2h^2q_i)y_1 + (2 - hp_1)y_2 = 2h^2r_1 - (2 + hp_1)a_0$

i=2,3...n-2

$(2 + hp_i)yi - 1 - (4 + 2h^2q_i)y_i + (2 - hp_i)y_{i+1} = 2h^2r_i$

i=n-1 $(2 + hp_{n-1})y_{n-2} - (4 + 2h^2q_{n-1})y_{n-1} = 2h^2r_{n-1} - (2 - hp_{n-1})b_0$

EXAMPLE:

Using Finite difference method to solve the following boundary value problem

y" = y + x(x-4) $0 \leq x \leq 4$

y(0) =y(4) =0

SOLUTION

Let h=1 , i.e, n=4

$y"(x_i) \approx \frac{y_{i+1}-2y_i+y_{i-1}}{h^2} = y_i + x_i(x_i - 4)$ **where i=1,2,3...**

$y'(x_i) \approx \frac{(y_{i+1}-y_{i-1})}{h}$

x_1 **= 0+h=0+1 =1**

x_2**= 0+2h=0+ 2 =2**

x_3 **=0+ 3h = 0+3 =3**

x_4**= 0+ 4h =0+4 =4**

With y(0) =0 , y(4) =0

i = 1 , $\frac{(y_2-2y_1+y_0)}{h^2} = y_1 + (x_1((x_1) - 4)$

$\implies \frac{(y_2-2y_1+0)}{1^2} = y_1 + 1(1-4)$

$\implies y_2 - 2y_1 = y_1 - 3$

$\implies -3y_1 + y_2 = -3$ ——————————————-**(1)**

i = 2, $\frac{(y_3-2y_2+y_1)}{h^2} = y_2 + (x_2((x_2) - 4)$

$\implies (y_3 - 2y_2 + y_1) = y_2 + 2(2-4)$

$\implies y_1 - 3y_2 + y_3 = -4$ ————————————-**(2)**

i = 3 , $\frac{(y_4-2y_3+y_2)}{h^2} = y_3 + (x_3((x_3) - 4)$

$\implies 0 - 3y_3 + y_2 = 3(-1)$

$\implies y_2 - 3y_3 = -3$ ———————————**(3)**

$$-3y_1 + y_2 = -3$$
$$y_1 - 3y_2 + y_3 = -4$$
$$y_2 - 3y_3 = -3$$

Which is a tri- diagonal system

Proceeding with Gaussian elimination method, the tri- diagonal system can be solved by

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & -3 \\ 1 & -3 & 1 & -4 \\ 0 & 1 & -3 & -3 \end{array}\right]$$

$R_1 \to \frac{R_1}{3}$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & 0 & 1 \\ 1 & -3 & 1 & -4 \\ 0 & 1 & -3 & -3 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & 0 & 1 \\ 0 & -8/3 & 1 & -5 \\ 0 & 1 & -3 & -3 \end{array}\right]$$

$R_2 \to -3/8R_2$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & 0 & 1 \\ 0 & 1 & -3/8 & 15/8 \\ 0 & 1 & -3 & -3 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & 0 & 1 \\ 0 & 1 & -3/8 & 15/8 \\ 0 & 0 & -21/8 & -39/8 \end{array}\right]$$

$R_3 \to -8/21R_3$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & 0 & 1 \\ 0 & 1 & -3/8 & 15/8 \\ 0 & 0 & 1 & 39/21 \end{array}\right]$$

$$\implies \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & -3/8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 15//8 \\ 39/21 \end{bmatrix}$$

$y_1 - \frac{y_2}{3} = 1$

$y_2 - \frac{3y_3}{8} = \frac{15}{8}$

$y_3 = \frac{39}{29}$

By back substitution we get the values as

$y_1 = \frac{13}{7}$

$y_2 = \frac{18}{7}$

$y_3 = \frac{13}{7}$

Chapter 3

STURM - LIOUVILLE PROBLEM

3.1 ORDINARY DIFFERENTIAL EQUATIONS

An ordinary differential equation is a differential equation in which there are one or more independent variables and one or more derivatives of a dependent variable with respect to independent variables . For example:

$y^{'} = sinx$,

$y^{''} + 10y^{'} + 9y = 0$

Consider an ordinary differential equation of second order,

$a_0(x)y^{''} + a_1(x)y^{'} + a_2(x)y = f(x)$

where f(x) is a function in x and x is an independent variable and y is a depentent variable ,then such a linear differential equation is said to be Non homogeneous differential equation . For example :

$y^{''} + y^{'} - 7y = 8x,$

$y^{'''} + 5y^{''} - y^{'} + y = 5e^{x}$

If f(x)=0 then the linear differential equation is said to be Homogeneous differential equation. For example :

$y^{''} + y^{'} - 7y = 0$

$y^{'''} + 5y^{''} - y^{'} + y = 0$

3.2 STURM LIOUVILLE PROBLEM WITH HOMOGENOEOUS BOUNDARY CONDITIONS

A Sturm – Liouville problem is a second order differential equation of the form:

$(p(x)y^{'})^{'} - q(x) + \lambda r(x)y = 0, a < x < b$

Together with the homogeneous boundary conditions:

$a_1 y(0) + a_2 y^{'}(0) = 0$

$b_1 y(1) + b_2 y^{'}(1) = 0$

Such an equation is said to be in Sturm-Liouville Problem. Here p, q and r are specific functions and λ is a parameter.Here a and b are not both zero. The solution of Sturm – Liouville problem consists of finding the values for λ for which the corresponding solution must be nontrivial and satisfying the boundary conditions. When such value of λ exists, they are called Eigen values and the corresponding solutions are called Eigen functions. A Sturm - Liouville problem is called regular if the functions q(x) and r(x) are continuous on [a,b] and also p(x) is differentiable on the same interval. Many Ordinary Differential Equations that occur during separation of variables can be put in the Sturm-Liouville form. For example, Bessel equation, Legendre's Differential equation, Chebyshev's Differential equation etc. can be reduced to this form. It's importance lies in its ability to provide an orthogonal basis for solving partial differential equations.

Boundary conditions are of the following forms:

- **Dirichlet condition :**$a_1 = b_1 = 1$ **and all other coefficients = 0;y(0) =0,y(1) =0**
- **Neumann condition :** $a_2 = b_2 = 1$ **and all other coefficients = 0;y'(0)=0,y'(1) =0**

PROPERTIES OF REGULAR STURM- LIOUVILLE PROBLEMS

- **There exist an infinite number of real eigen values that can be arranged in increasing order** $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$ **such that** $\lambda_n \to \infty$

as n $\rightarrow \infty$

- For each eigen value there is only one eigen function (except for constant multiples).
- Eigen functions corresponding to different eigen values are linearly independent.
- The set of eigen functions corresponding to the set of eigen values is orthoganal with respect to the weight function p(x) on the interval [a,b]

3.3 REDUCTION TO STURM- LIOUVILLE EQUATIONS

All homogeneous second-order linear ordinary differential equations can be reduced to Sturm – Liouville Equation . All second-order linear ordinary differential equations can be converted in the form of Sturm-Liouville Equation by multiplying both sides of the equation by an appropriate integrating factor. This helps us to solve many Ordinary differential equations for easy calculation and to reduce the complexity of Homogenous Differential Equations. Given below are some of the examples:

- BESSEL'S EQUATION TO STURM- LIOVILLE EQUATION
  $x^2y^{"} + xy^{'} + (x^2 – v^2) = 0$ where v is a parameter
  This is a Bessel equation. This can be converted to Sturm–Liouville form by dividing throughout by x, then by collapsing the first two terms on the left into one term.
  Example: $x^2y^{"} + xy^{'} + (\lambda^2x^2 - m^2)y = 0$
  This is a parametric Bessel equation which is converted to Sturm-Liouville form.
  First we divide by x to get,
  $xy^{"} + y' + (\lambda^2x - \frac{m^2}{x})y = 0$

  This is in Sturm-Liouville form with
  $p(x) = x$, $q(x) = \frac{-m^2}{x}$, $r(x) = x$,
  provided we write the parameter as λ^2.

- **LEGENDRE'S EQUATION TO STURM- LIOVILLE EQUATION**

  The Legendre equation given by,

  $(1-x^2)y^{"} - 2xy^{'} + v(v+1)y = 0$

  this can easily be put into Sturm–Liouville form, since

  $\frac{d}{dx}(1-x^2) = -2x,$

  so the Legendre equation is equivalent to

  $((1-x^2)y^{'}))' + v(v+1)y = 0$

  For example: Consider the Legendre equation,

  $y^{"} - (\frac{2x}{1-x^2})y^{'} + (\mu/1-x^2)y = 0$

  we multiply by $1-x^2$ **to get,**

  $(1-x^2)y^{"} - 2xy^{'} + \mu y = 0$

  This is in Sturm-Liouville form with

  $p(x) = 1-x^2$, $q(x) = 0$, $r(x) = 1$, **provided we write the parameter as** μ.

- **CHEBYSHEV'S DIFFERENTIAL EQUATION TO STURM- LIOVILLE EQUATION**

  Consider the Chebyshev's differential equation

  $(1-x^2)y^{"} - xy' + n^2y = 0$

  we divide by $\sqrt{1-x^2}$ **to get**

  $(\sqrt{1-x^2})y^{"} + (\frac{-x}{\sqrt{1-x^2}})y + (\frac{n^2}{\sqrt{1-x^2}})y = 0$

  This is in Sturm-Liouville form with $p(x) = \sqrt{(1-x^2)}$, $q(x) = 0$, $r(x) = \frac{1}{\sqrt{1-x^2}}$,

  provided we write the parameter as n^2.

- **CONVERTING TO STURM- LIOUVILLE EQUATION USING INTEGRATING FACTOR**

  A second order differential equation can be converted to Sturm–Liouville equation by first, dividing the differential equation by the coefficient of $y^{"}$ **and multiplying throughout by integrating factor**

  $\mu(x) = e^{(\int \frac{-dx}{x^2})}$

  Example: $x^3y^{"} - xy^{'} + 2y = 0$

  Divide throughout by x^3**:**

  $y^{"} - (\frac{1}{x^3})y^{'} + (\frac{2}{x^3})y = 0$

Multiplying throughout by an integrating factor of
$\mu(x) = e^{(\int \frac{-dx}{x^2})} = e^{\frac{1}{x}}$
gives;

$(e^{\frac{1}{x}})y" - (\frac{e^{\frac{1}{x}}}{x^3})y' + (\frac{2e^{\frac{1}{x}}}{x})y = 0$
which can be easily put into Sturm–Liouville form since,

$\frac{d}{dx}(e^{\frac{1}{x}}) = \frac{-(e^{\frac{1}{x}})}{x^2}$
so the differential equation is equivalent to

$((e^{\frac{1}{x}})y')' + (\frac{2e^{\frac{1}{x}}}{x^3})y = 0.$

3.4 SOLVING EIGEN VALUES AND EIGEN FUNCTIONS

To solve an eigen value problem the following steps are used:
The differential equation is solved as an initial-value problem for a series of trial values of λ, modified until the boundary conditions are satisfied simultaneously at both ends, yielding eigenvalues at such λ .The simplest shooting technique is to choose an initial condition that satisfies one of the boundary conditions, then shoot from that endpoint (by varying λ) until there are no solutions at the other endpoint. For this first we consider a boundary value problem which consists of a second order differential equation of the form
$(p(x)y')' - q(x) + \lambda r(x)y = 0$
where p ,q,r are real functions such that has a continuous derivative ,q and r are continuous and $p(x) > 0$ and $r(x) > 0$ for all x on an interval [a,b] and λ is a parameter independent of x. Then we consider three cases $\lambda = 0$, $\lambda < 0$, $\lambda > 0$. In each case we find the general solution of the differential equation and determine the arbitrary constant in each solution using the boundary conditions . when $\lambda = 0$ and $\lambda < 0$ there are no non trivial solution and when $\lambda > 0$ we get a non trivial solution satisfying the boundary conditions ,where the value of λ is the eigen value or characteristic value and the function defined is called the eigen

function or characteristic function . Examples:

1)Solve the Eigen value problem

$$y" + 3y' + 2y + \lambda y = 0 \tag{3.1}$$

y(0)=0,y(1)=0

SOLUTION

The characteristic equation of the 3.1 is given by

$m^2 + 3m + 2 + \lambda = 0$

solving the above equation gives the solution,

$m_1 = -3 + \sqrt{1 - 4\lambda/2}$ **and** $m_2 = -3 - \sqrt{1 - 4\lambda/2}$

case 1:If $\lambda <$ **then** m_1 **and** m_2 **are real and distinct so the general solution of the differential equation in 3.1 is**

$y = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

The boundary conditions require that $c_1 + c_2 = 0$

$c_1 e^{m_1} + c_2 e^{m_2} = 0$ **Since the determinant of this system is** $e^{m_2} - e^{m_1} \neq 0$, **the system has only the trivial solution. Therefore** λ **is not an Eigen value of 3.1 .**

Case 2:If λ**=**$\frac{1}{4}$ **then** m_1=m_2**=-3/2,so the general solution of the given problem is** $y = e^{-3x/2}(c_1 + c_2 x)$

The boundary condition y(0)=0 requires that $c_1 = 0$**,so** $y = c_2 x e^{-3x/2}$ **and the boundary condition y(0) requires that** $c_2 = 0$. **Therefore** λ**=**$\frac{1}{4}$ **is not an Eigen value of 3.1 .**

Case 3: If $\lambda >$ **then** $m_1 = -3/2 + i\omega$ **and** $m_2 = -3/2 - i\omega$ **with**

$\omega = (\sqrt{4\lambda - 1})/2$ **or**

$$\lambda = (1 + 4\omega^2)/4 \tag{3.2}$$

In this case general solution of differential equation in (1) is $y = e^{-3x/2}(c_1 cos\omega x + c_2 sin\omega x)$ **The boundary condition y(0)=0 requires that** $c_1 = 0$,

so $y = c_2 e^{-3x/2} sin\omega x$ **which holds with** $c_2 \neq 0$ **if and only if** $\omega = n\pi$, **where n is an integer we may assume that n is a positive integer . From 3.2, the Eigen values are**

$\lambda_n = (1 + 4n^2\pi^2)/4$

with associated Eigen functions $y_n = e^{-3x/2} sinn\pi x$**; n=1,2,3,...**

2)Solve the Eigen value problem

$$y'' + \lambda y = 0 \tag{3.3}$$

y(0)=0; y(π)=0

SOLUTION

Case 1: λ**=0 In this case 3.3 reduces as y"=o .**

Therefore the general solution is y=$c_1 + c_2x$

By applying the boundary conditions : First applying y(0)=0, we obtain c_1**=0**

Applying Y(π**)=0,we get** $c_1 + c_2\pi$ **=0.since** c_1**=0 then** c_2**=0**

$\implies c_1 = c_2$**=0**

But the solution becomes y(x)=0 for all values of x.

For λ**=0/the only solution of the given problem is the trivial solution.**

Case 2: $\lambda < 0$

The charateristic equation of 3.3 is $m^2 + \lambda$**=0 and we get the roots as** $\pm\sqrt{(-\lambda)}$**.**

These roots are real and unequal we denote $\sqrt{(-\lambda)}$ **by** α**. Then for** $\lambda < 0$ **the general solution will be**

$y = c_1e^{\alpha x} + c_2e^{-\alpha x}$

By applying the first boundary condition y(0)=0,we get $c_1 + c_2$**=0**

Applying the second condition y(π**)=0,we get**

$c_1e^{\alpha\pi} = c_2e^{-\alpha\pi} = 0$

Here also $c_1 = c_2 = 0$ **which gives only the trivial solution . When we find the determinant**

$\det\begin{pmatrix} 1 & 1 \\ e^{\alpha\pi} & e^{-\alpha\pi} \end{pmatrix}$ **as** $e^{\alpha\pi} = e^{-\alpha\pi}$ **and To get a nontrivial function** α**=0 but here** $\alpha = \sqrt{(-\lambda)}$**.There are no non trivial solutions when** $\lambda < 0$.

Case 3:$\alpha > 0$ **Here the roots are** $\pm\sqrt{(-\lambda)}$ **are the conjugate complex numbers** $\pm\sqrt{(\lambda i)}$ **.Therefore the general solution will be**

$y = c_1 sin\sqrt{(\lambda x)} + c_2 cos\sqrt{(\lambda x)}$

Applying first boundary condition we get

$c_1 sin0 + c_2 cos0 = 0$

Applying second condition ,

$c_1 sin\sqrt(\lambda\pi) + c_2 cos\sqrt(\lambda\pi)$**=0**

since $c_2 = 0 \implies c_1 sin\sqrt(\lambda\pi) = 0$

here either we can get c_1**=0 or** $sin\sqrt(\lambda\pi)$**=0;If** $c_1 = 0$ **then** c_2**=0 ,we get an unwanted trivial solution .Thus to obtain a nontrivial solution**

$sin\sqrt(\lambda\pi) = 0$

sin (kπ)=0,if $k > 0$ **that is n=1,2,3...**

Here $\sqrt{\lambda}$**=n,where n=1,2,3...**

$\implies$ **Eigen function :**$y = c_n sinnx$ **,where n=1,2,3...**

Eigen value :$\lambda = n^2$**, where n=1,2,3...**

Chapter 4

APPLICATIONS OF STURM LIOUVILLE PROBLEM

4.1 SOUND WAVE ANALYSIS

Sound wave analysis is the process of using mathematics to study and analyse the behavior of sound waves. Sturm-Liouville Problem can be used to analyse sound waves and it helps in the calculation of the frequencies of various notes. This probem depends on the set of differential equations and boundary conditions that define the behavior of a wave. By solving these equations, the frequencies at which the wave oscillates can be determined. Sturm-Liouville Problems involve solving a standard differential equation for a standard Simple Harmonic Oscillator of the form,

$\frac{d^2y}{dx^2} + ky = 0$

where k is a constant that determines the behavior of the wave. The boundary conditions for the problem are also important; they define the starting and ending points of the wave, as well as any other restrictions on the wave's behavior. Once the equation has been solved, the frequencies at which the wave oscillates can be determined. This analysis can be used to determine the frequency of a given note, as well as the frequencies of any overtones or harmonics associated with that note .

4.2 THE STRUCTURAL STABILITY OF BRIDGES

These problems are used to analyse the effects of external forces on the bridge and its components. The aim of these problems is to determine the amount of force that a bridge can resist before it begins to deform, buckle, or collapse. This analysis is important for engineers to ensure that bridges are safe . Sturm-Liouville problems involve the use of differential equations to model the behavior of a bridge. These equations incorporate the effects of external forces, such as wind and traffic, as well as the properties of the bridge materials. The equations are used to determine the static and dynamic responses of the bridge to these external forces. The equations also provide information on the amount of force that can be applied to the bridge before it begins to deform or collapse. The solutions to these problems are used to evaluate the structural stability of the bridge. The analysis also helps to identify any weak points in the design of the bridge. This information can be used to make improvements in the design of the bridge .

4.3 QUANTUM MECHANICS

Sturm Liouville problems are used to solve quantum mechanical equations. They are used to describe the behavior of a quantum state in terms of its energy, momentum, position, and spin. These equations can be used to calculate the energies of different quantum states and the probabilities of different transitions between them. They can also be used to determine the wave functions of a given state. Sturm Liouville problems are often encountered in the study of quantum mechanics. The equations are used to solve problems in quantum mechanics such as solving the Schrödinger equation with particular boundary conditions or to calculate the energy spectrum of a quantum system. They are also used in quantum field theory and quantum information theory.

4.4 IMAGE PROCESSING

Image processing is an important process for analyzing and manipulating digital images. One of the most widely used tools for image processing is the Sturm-Liouville Problem. This problem is used to analyze digital images and calculate the contrast and brightness of various points within the image. These equation can be solved using boundary conditions to measure the brightness and contrast of a certain point in the image. The equation is solved by finding the eigenvalues and eigenfunctions of the equation. These values are then used to calculate the contrast and brightness of each point in the image, allowing for a more detailed analysis of the image. This process can also be used for more complex images, such as facial recognition. The equation can be used to measure the contrast and brightness of a person's face, helping to identify them.

4.5 ELECTRICAL NETWORKS

The Sturm-Liouville problem is a fundamental tool used in the analysis of electrical networks, and it can be used to solve a variety of electrical network problems. Specifically, it can be used to find the natural modes of vibration of a network, the impedance(It is the effective resistance of an electrical circuit to apply AC) and admittance(It is the measure of electrical conduction) matrices of a network, the impedance and admittance of a single element in a network, and the transfer functions of a network. Furthermore, it can be used to solve the inverse problem of finding the element values of a given network from its transfer functions. Finally, it can be used to solve problems involving the design of linear electrical networks, such as filters and amplifiers. Sturm Liouville problems have applications in AC and DC circuits. In AC circuit analysis, they can be used to solve differential equations describing the behavior of the circuit and the response of the system to various inputs. In DC circuits, they can be used to solve for the steady-state operating conditions of the circuit. It is an important tool in the study of linear systems and is used to study the properties of eigenvalues, eigenfunctions, and the spectral decomposition of a system. The problem is used to determine the natural frequencies of oscillation of a system, which are important for understanding the characteristics of a circuit. In addition, It can be used to calculate the transfer functions of circuits and the energy of a system. This is especially useful in the analysis of AC and DC circuits, as it allows for the determination of the total power present in a circuit and the amount of power dissipated as heat. This can be used to optimize the design of electrical and mechanical systems

4.6 TELECOMMUNICATIONS

In telecommunications, the Sturm - Liouville problem is used to model and analyze wireless communication networks. These networks consist of multiple nodes (e.g., base stations and mobile phones) connected by wireless links. The goal of these networks is to provide reliable and efficient communication between the nodes. To do this, the network must be able to determine the best paths for transmitting data across the network. The Sturm - Liouville problem is used to solve this problem by finding the eigenvalues and eigenfunctions of the network. These eigenvalues and eigenfunctions describe the characteristics of the network, such as the signal strength and propagation delay of the different paths. By using these eigenvalues and eigenfunctions, the network can be optimized to provide the best communication service. It can be used to analyze the capacity of a network. Capacity is the maximum amount of data that can be transmitted through the network without loss or delay. The eigenvalues and eigenfunctions of the network can be used to calculate the capacity of the network and thus optimize it for maximum performance. The Sturm-Liouville problem is of great importance in telecommunications due to its ability to analyze transmission line systems. By this problem, engineers can determine the electrical characteristics of a transmission line, such as its propagation speed, attenuation and power loss. This information can then be used to design and optimize telecommunications networks. Additionally, the Sturm-

Liouville problem can be used to analyze and design various types of antennas, which are essential components in telecommunications systems.

4.7 ACOUSTICS WAVES

Acoustic waves are used to study the behavior of sound waves in a variety of media, including air and water which travel through space and they are essential for many different processes such as sound production and communication. Sturm-Liouville theory is used to analyze the properties of these waves, such as their frequency, amplitude, and propagation speed.. The frequency of a system is determined by the eigenvalues of the equation, while the energy is determined by the eigenfunctions. This allows the behavior of the system to be described in terms of its eigenfunctions and eigenvalues, which can be used to predict the behavior of the system under different conditions.By solving the equation, it is possible to determine the speed at which a sound wave travels through a medium, which can be used to design sound systems that are optimized for specific environments. It can also be used to reduce the amount of distortion in a sound system, as well as to improve the clarity of the sound. They are also used to analyze the behavior of acoustic resonators, such as drums and guitar strings. These problems can help us understand why certain frequencies resonate better than others, and can also be used to design acoustic systems that can control and manipulate sound waves.The direct Sturm -Liouville Problem is used for analysing surface love waves propagating in layered viscoelastic waveguides. The love waves are surface waves that move parallel to the Earth's surface and perpendicuar to the direction of propagation.

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