Project Report

On

APPLICATION OF EIGENVALUES AND EIGENVECTORS

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by

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CERTIFICATE

This is to certify that the dissertation entitled, **APPLICATION OF EIGENVALUES AND EIGENVECTORS** is a bonafide record of the work done by Ms. **NEHA SREEHARI** under my guidance as partial fulfillment of the award of the degree of **Bachelor of Science in Mathematics** at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of DR ELIZABETH RESHMA M T, Assistant Professor, Department of Mathematics, St. Teresa's College(Autonomous), Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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Date:28/02/2023 **AB20BMAT044**

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I must mention several individuals who encouraged me to carry this work. Their continuous invaluable knowledgably guidance throughout the course of this study helped me to complete the work up to this stage

I am very grateful to our project guide Dr. Elizabeth Reshma MT for her guidance and support throughout the work. I also want to express my gratitude to my classmates, who assisted me in formulating my research topics by sharing their knowledge. Last but not the least, I would like to express my heartfelt gratitude to God Almighty for always being my emotional rock and teaching me how to confront every challenge with intelligence and confidence.

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Chapter 1

INTRODUCTION AND PRELIMINARIES

1.1 INTRODUCTION

Eigenvalues and eigenvectors are one of the important concepts in the study of Linear Algebra. Eigenvalues are associated with eigenvectors and both are very much used in the analysis of linear transformations. Eigenvalues are the special set of scales associated with the matrix representation of system of linear equations. Eigenvectors are non zero vectors that corresponds to each eigen value, whose direction remain unchanged when a linear transformation is applied to it. Eigenvalues and eigenvectors plays an important part in data science and various other fields of mathematics, engineering and science. In this project we deal with two of the important applications of eigenvalues and eigenvectors namely in Face Recognition and in solving nth order homogeneous linear differential equations. We will also see their corresponding applications in various areas of science and machine learning. The word 'eigen' means 'self' in modern German. The eigenvalues and eigenvectors for a linear transformation were found in matrix algebra and was first invented in mid 19th century by English mathematician Arthur Cayley .

1.2 HISTORY

Johann and Daniel Bernoulli, D' Alembert, and Euler come across eigenvalue problems in the first half of the 18th century when they were studying about the motion of a rope, which they considered as a weightless string loaded with a number of masses. In the second half of the 18th century, Laplace and Lagrange continued their study on eigenvalues. They realised that eigenvalues are related to the stability of the motion. Eigenvalue method is also used in the study of solar system by Laplace and Lagrange. Towards the end of the 19th century, Schwartz studied the first eigenvalue of Laplace's equation on general domain.The words " eigenvector" and "eigenfunction" were not standardised words until into 20th century. All kinds of words like proper vectors, characteristic vectors were used. At the beginning of the 20th century, Hilbert studied about the eigenvalues of integral operators by considering them to be infinite matrices. Hilbert was the first to use the German word Eigen to denote eigenvalues and eigenvectors in 1904, though he may have been following a related usage by Helmholtz. The first numerical algorithm for computing eigenvalues and eigenvectors appeared in 1929.

1.3 PRELIMINARIES

DEFINITIONS

Definition 1:(Eigenspace)

An eigenspace is the collection of eigenvectors of each eigenvalues for the transformation applied to eigenvector. The linear transformation is often a square matrix.

Definition 2:(Characteristic polynomial)

The characteristic polynomial of a square matrix is an invariant polynomial under matrix similarity and has roots as eigenvalues. The characteristic equation is the equation obtained when equating the characteristic polynomial to zero.

Definition 3: (Similar matrices)

Two square matrices are similar if they represent same linear operator under different bases. Two similar matrices have same rank, determinant and eigenvalues.

Definition 4: (Row equivalent matrices)

Two matrices are said to be row equivalent if one can be changed to other by a sequence of elementary row operations.

Definition 5:(Eigen face)

This method is useful for face recognition and detection by determining the variance of faces in a collection of face images and use those variances to encode and decode a face in a machine learning way.

Definition 6:(Normalised eigenvector)

Normalized eigenvector is nothing but an eigenvector having unit length.

Definition 7:(Characteristic equation)

The characteristic equation of a square matrix A is $|A - \lambda I| = 0$ **where** λ **is any scalar.**

Definition 8:(Hermitian matrix)

A square matrix A is said to be Hermitian matrix if $A^T = A$. **The Hermitian matrix always have real eigen values.**

Definition 9:(Unitary matrix)

If a square matrix A satisfies the condition $A^T = A^{-1}$, **then A is said**

to be a unitary matrix. If all the entries of unitary matrix are real numbers, then it is said to be the orthogonal matrix.

Definition 10:(Covariance matrix)

Covariance matrix is a square matrix that shows the variance exhibited by elements of datasets and the covariance between a pair of datasets.

Definition 11:(Orthogonal matrix)

If $A.A^T = I$ **for a square matrix with all real entries, then A is called Othogonal matrix .**

PROPERTIES

Property 1:

The eigenvectors of a matrix must be non zero but an eigenvalue may be 0.

Property 2:

The eigenvalue of a triangular matrix are the entries on its main diagonal so the determinant equals to the product of its eigen values

Property 3:

If $v_1, \ldots\ldots, v_m$ **are eigenvectors that corresponds to distinct eigen values** $\lambda_1, \ldots\ldots, \lambda_m$**, of an n x n matrix A then the set** $\{v_1, \ldots.., v_m\}$ **is linearly independent.**

Property 4:

If an n x n matrix has distinct eigen value then the matrix is diagonalisable.

Property 5:

A Hamilton matrices and symmetric matrices has all the eigen values real numbers.

Property 6:

If $\lambda_1, \lambda_2, ..., \lambda_n$ **are the eigen values of matrix A, then**

1. **eigen values of** A^{-1}**are** $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ...\frac{1}{\lambda_n}$

2. **eigen values of** A^k **are** λ_1^k**,** λ_2^k**,...,**λ_n^k

ILLUSTRATION USING PYTHON CODE

Let A be an n x n matrix, then λ belongs to R is called an eigen value of A if there exist a non-zero vector x in R such that $Ax = \lambda x$
The vector x is called an eigen vector for the corresponding λ

2 x 2 matrix

Let us find the characteristic polynomial of $A =$

$$\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$$

Now $|A - \lambda I_2| = (1 - \lambda)(5 - \lambda) + 3 = 5 - \lambda - 5\lambda + \lambda^2 + 3 = \lambda^2 - 6\lambda + 8$

On solving the characteristic polynomial we get the eigen values.
i.e., $\lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$
i.e., $\lambda = 4$ **or**$\lambda = 2$
The eigen values of A are 4 and 2.
The corresponding eigenvectors are found by using these λ values in the equation$(A - \lambda I_2)x = 0$
$\lambda = 4$
We solve $(A - 4I_2)x = 0$

$$\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives the system of equations,
$R_2 \implies R_2 + R_1$

$$\begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 - x_2 = 0$
i.e., $-3x_1 = x_2$
i.e.,$x_2 = k$ **and**
$x_1 = -3k$
Thus Eigen vectors for $\lambda = 4$ is

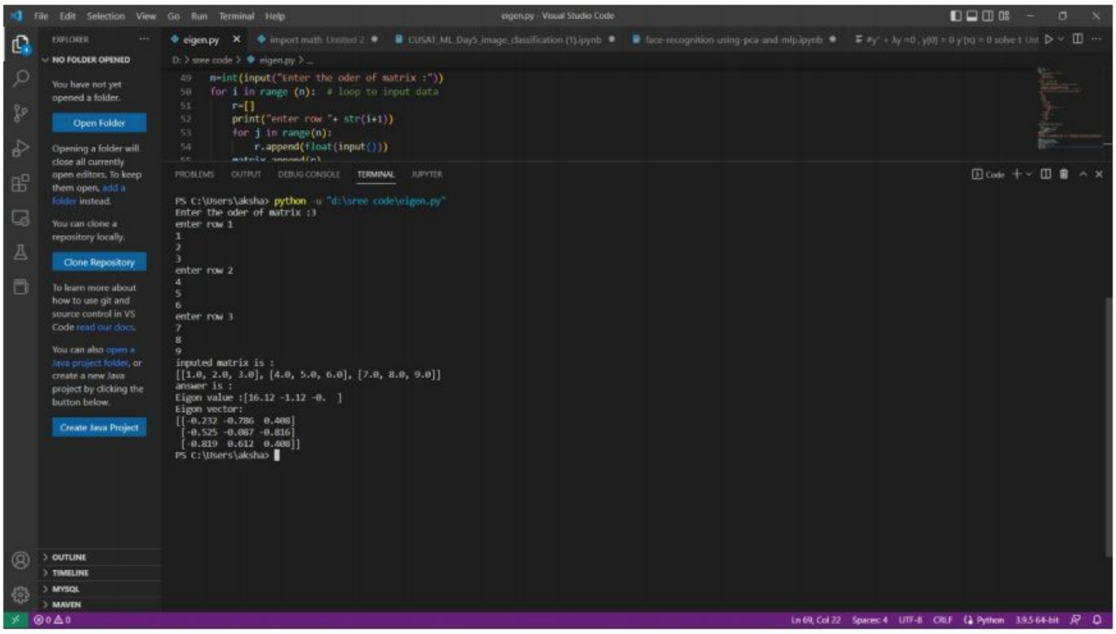

Figure 1.1: Output of the program for finding eigen values and eigen vectors of n×n order matrix.

$$k\begin{bmatrix}-3\\1\end{bmatrix}$$

$For \lambda = 2$

$$\begin{bmatrix}-1 & -1\\3 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$R_2 \implies R_2 + 3R_1$

$$\begin{bmatrix}-1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

i.e., $-x_1 - x_2 = 0$

i.e., $-x_1 = x_2 = u$

i.e., $x_2 = u$ **and**

$x_1 = -u$

$$u\begin{bmatrix}-1\\1\end{bmatrix}$$

Therefore, the eigen values are $\lambda_1 = 4$ **and** $\lambda_2 = 2$ **and the corresponding eigen vectors are** $v_1 = \begin{bmatrix}-3\\1\end{bmatrix}$ **and** $v_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

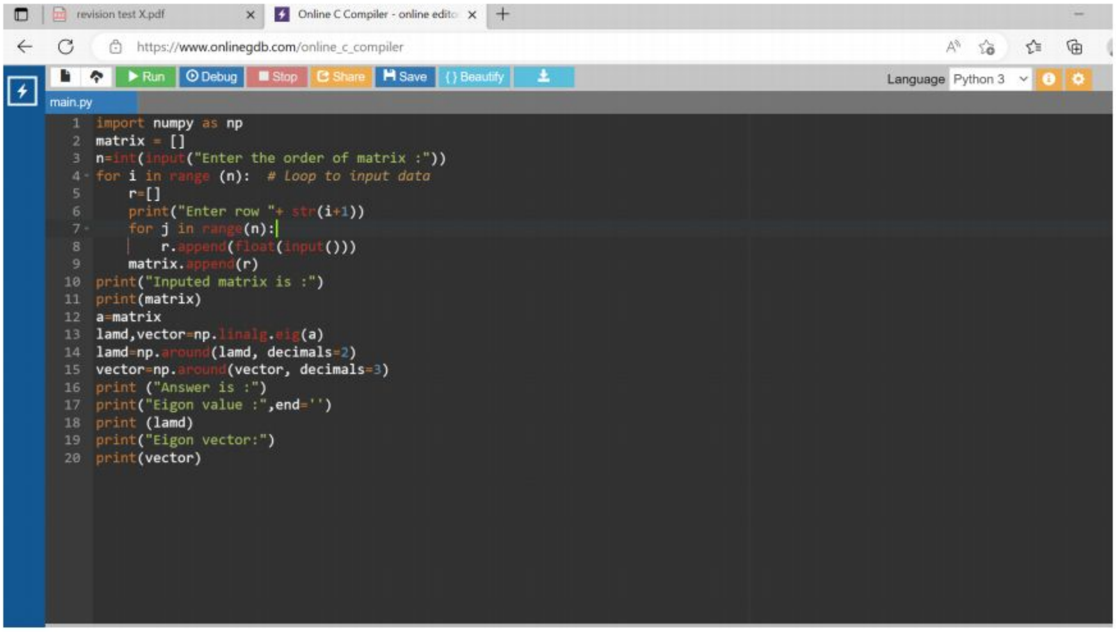

Figure 1.2: Output of the program for finding eigen values and eigen vectors of 2x2 order matrix.

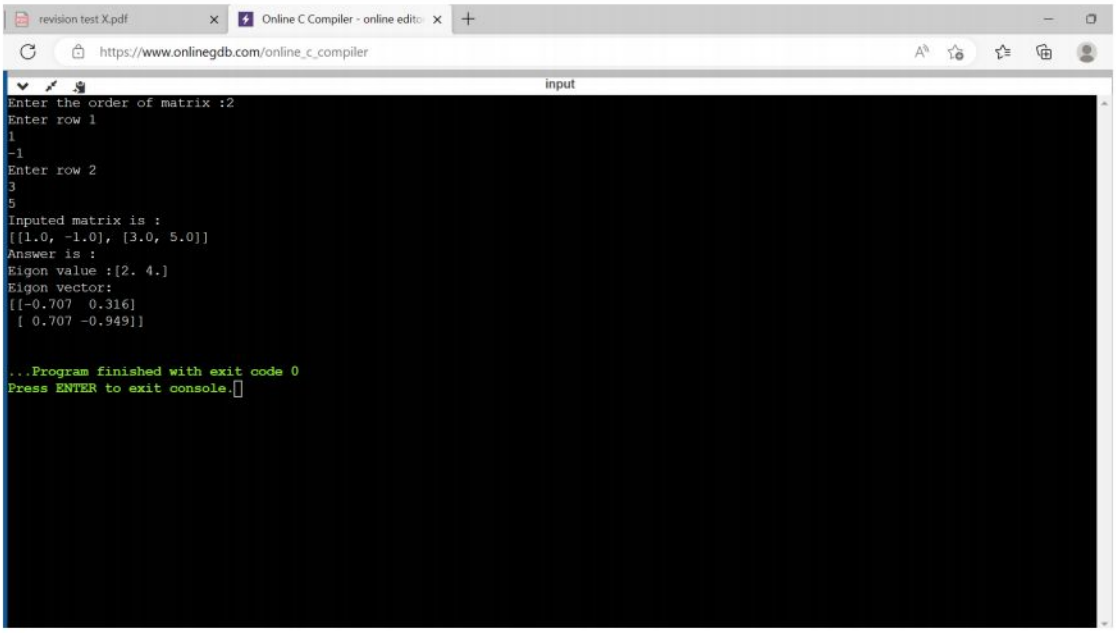

Figure 1.3: Output of 2x2 order matrix.

CAYLEY HAMILTON THEOREM

The theorem named after the mathematician Arthur Cayley and Witham Rowan Hamilton is one of the best known properties of characteristic polynomial. It states that every square matrix will satisfy its characteristic equation over a commutative ring of real or complex field that is if the characteristic equation of an nxn matrix A is $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0$

Then,$A^n + a_{n-1}A^{n-1} + ... + a_1A + a_0I = 0$

1.4 SOME APPLICATIONS OF EIGEN VALUES AND EIGEN VECTORS

BRIDGE DESIGNING

The Eigen value of the smallest magnitude of a system which models the bridge describes the natural frequency of the bridge. This knowledge is used by the engineers to ensure stability of bridges during their construction. The importance of this analysis can be seen from the collapse of Tacoma Narrows Bridge, Washington which was opened to public on July 1940 but collapsed into Puget Sound on November 7 the same year. It collapsed because the main span that already marked flexibility went to a series of torsional oscillations finally increased the amplitude of oscillations steadily and the span broke. Hence eigen values are required to know the natural frequency of the bridges and hence ensure stability.

WEATHER FORECASTING IN BELFAST

A matrix model was developed for the weather in Belfast, Northern Island and is explained in a book ' Introduction to Numerical solution of Markov chains' by William Stuart. Three weather conditions are mainly considered : Rainy(R), Cloudy (C), and Sunny (S). Daily weather changes are described by the matrix.

MACHINE LEARNING

Eigen values and eigen vectors finds a wide area of usage in Machine learning for dimensionality reduction which is important in face recognition, image processing, etc. Tasks which are highly computationally

exhaustive can be reduced to ones with lower dimensions with a tool called PCA (Principle Component Analysis) which reduces dimensions of multivariate set to smaller set that contains most of the information of the larger set with loss of very little information. Eigen values and its corresponding vectors are the basis for PCA.

COMMUNICATION SYSTEM

Claude Shannon used eigenvalues to define the theoretical limit of how much information can be carried through a chanel of communication like air or telephone line. The eigenvalues and eigenvectors of the channel of communication represented in matrix form is calculated, then the eigenvalues are waterfilled. The essential gains of the channel's fundamental modes are essentially eigenvalues, which are recorded by the eigenvectors.

Here we have discussed some of the applications of eigen values and eigen vectors. In the following chapters we discuss two of the main applications that is application in Face Recognition and in solving nth order homogeneous linear ordinary differential equations.

Chapter 2

HUMAN FACIAL RECOGNITION

Facial recognition is a technology capable of identifying and matching a human face through an image or video. This technology is often used in biometric security systems and ID verification services by analyzing the facial features from the given image. This technology collects biometric data for each person associated with their facial features and facial expression to identify, analyze and match the identity of a person.

Based on the characteristics of face, face recognition can be divided into two groups:

1. Appearance based, which uses integrated texture features and is applied to either whole face or specific regions in an image of space.
2. Feature based, which uses geometric facial features like for example mouth, colour or size of the eyes, cheeks, size of the brows etc.

The use of identification and authentication in people's daily lives has become more frequent with the development of information technology and networks. For the current identification requirements, the traditional authentication methods based on user name and password are less suitable. Therefore, more convenient, reliable, and secure authentication methods are introduced. One such is the face recognition technology. Face recognition is a versatile application technology that has been developed with the increasing demand for information security. Face recognition has become the research object of many researchers

for the reason that it takes the human face as the object of recognition and meets the requirements of identification.
There are three major tasks in facial recognition system, they are

1. Loading the image

Loading the image is the first step in the face recognition system. You can obtain faces from a variety of available face database. The main requirements are that the faces images must be Greyscale (Grayscale is a range of shades of gray without apparent color) image with a consistent resolution. If we are having a colour images, then first they should be converted into grayscale image. If the image includes the background, the face recognition will not work properly, as the background will be incorporated into the classifier. So we have to crop the image only to show the face of the person. Also while loading, try to avoid hair since hair style of the person can change significantly (or if they wear a hat can also change the persons look).

2. Training

The following steps are required for training the face detector
a. Calculate the mean of the input face images.
b. Subtract the mean from the input images to obtain the mean-shifted images.
c. Calculate the eigenvectors and eigenvalues of the mean-shifted images.
d. Order the eigenvectors by their corresponding eigenvalues, in decreasing order.
e. Retain only the eigenvectors with the largest eigenvalues .
f. Project the mean-shifted images into the eigenspace using the retained eigenvector

3.Classification and matching

Classification and Matching:Once the face images have been projected into the eigenspace, the similarity between any pair of face images can

be calculated by finding the Euclidean distance between their corresponding feature vectors and ; the smaller the distance between the feature vectors, the more similar the faces. We can define a simple similarity score based on the inverse Euclidean distance.To perform face recognition, the similarity score is calculated between an input face image and each of the training images. The matched face is the one with the highest similarity, and the magnitude of the similarity score indicates the confidence of the match (with a unit value indicating an exact match).Given an input image with the same dimensions image as your training images, the following code will calculate the similarity score to each training image and display the best match.

STEPS IN FACE RECOGNITION

In general, the face recognition system has 4 steps as follows:

Face detection

The camera detects the image of the face to be analyzed. These technologies are capable of locating faces in crowded areas.

Face analysis

The aim is to identify the facial features and the key to distinguishing your face from others.

Creating data from the images

The image of the face captured is transformed into digital data regarding the facial features of that person. It is transformed into a mathematical formula called face print which is unique to each person.

Matching and recognizing

The face print thus developed is compared with a database of known faces. If the face print matches with the image in the database, we can determine the identity of that person.

2.1 ADVANTAGES AND DISADVANTAGES OF FACE RECOGNITION

ADVANTAGES OF FACE RECOGNITION

Increased security:

In governmental level it will help to find out terrorists and other criminals. In personal level it will help as a security for locking personal devices and personal surveillance cameras.

Greater convenience:

In this post COVID-19 world, face recognition plays a major role. This will help people to pay bills in markets without using credit cards and all, also it will help from avoiding big queues and help in saving time.

Faster processing:

Facial recognition is a quick and efficient process which helps big companies from cyber attacks and advanced hacking tool, companies need both secure and fast technologies.

Reduced crime:

The knowledge of the presence of face recognition system can serve as a deterrence, especially for petty crime. The companies can use face recognition to access computers to avoid misusage of data.The benefit is that there is nothing to change or steal in this case so it cannot be hacked.

Removing bias from stop and search:

Public concern over unjustified stops and searches is a source of controversy for the police — facial recognition technology could improve the process. By singling out suspects among crowds through an automated rather than human process, face recognition technology could help reduce potential bias and decrease stops and searches on law-abiding citizens.

DISADVANTAGES OF FACE RECOGNITION

Surveillance:

Facial recognition technology allows government to track down criminals but it could also allows to track down innocent and ordinary people at any time. So some people worry that the face recognition using ubiquitous video cameras, artificial intelligence and data analytics creates the potential for mass surveillance, which could restrict individual freedom.

Scope for error:

In 2018 Newsweek it is reported that Amazon's facial recognition technology has falsely recognized 28 members of US Congress as people arrested for crime, this shows that the facial recognition is not free from error it will also lead to miss interpretation of normal people which could make them criminals a slight change in people's face, like hair may lead to error in face recognition.
Breach of privacy individual needs privacy:
The government is known to store several citizen's picture without their consent, so the facial recognition may lead to loss privacy and ethical abuses.
Massive data storage:
The facial recognition is done using software relies on machine learning technology which requires massive data sets to learn to deliver accurate results. Such large data set requires robust data storage. Small and medium sized companies may not have sufficient resources to store the required data

2.2 APPLICATIONS OF FACE RECOGNITION TECHNOLOGY

Phone locks:
Various smartphones used face recognition technology as a way to unlock the device. It is one of the most powerful ways to protect personal data on your mobile phone.
Law enforcement:
Facial recognition is often used by law enforcement authorities. The culprit's photos are collected in the database then the authority can identify them.
In airports and borders:
Face recognition is a very well gives technology in airports. It not only helps to reduce the long lines but also improves security in airports.
Tracking missing people:
Facial recognition can be used to identify and track missing people. If we have a database added missing individuals we could recognize them

by facial recognition technology.

Reduce theft in shops:

The photographs of known shoplifters can be collected in a database of criminals so that security could identify them and can be notified when they enter the shop through the CCTV cameras.

Banking sector:

Facial recognition technology can be a great method to authorized transactions in the banking sector. It is more secure as passwords are not involved. Through the live less detection technique no hackers could use a photo to steal money from the account.

Marketing:

A lot of companies use facial recognition technology for the marketing of products.

Hospitals:

Facial recognition could help hospitals to identify patients' medical history. It was a good help in detecting emotion, pain, and even specific genetic diseases in patients.

Attendance:

using face recognition technology can be the best way of taking attendance. Some schools in China use face recognition to take attendance for the students. It can be also used check the signing in and out of the workers in a workplace.

Recognizing drivers in cars:

Some car companies are experimenting to use face recognition technology as a replacement for car keys. It reduces the theft of cars and could be an easy way to operate the car.

ILLUSTRATION

Obtain face images $I_1, I_2, I_3, ..., I_m$ as training dataset faces.

Consider m=4 and size of each image N x N

N = 2

Image matrix:

$$I_1 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 3 & 5 \\ -1 & 0 \end{bmatrix}$$

Represent every image I_i as a vector Γ_i

The image vector Γ_i is used thereafter in all steps of calculation

$$\Gamma_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix}$$

$$\Gamma_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\Gamma_4 = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

n = 2[order of the matrix]

m = 4[maximum number of image faces]

Mean face vector,

$$\psi = \frac{1}{M}\sum_{i=1}^{m}\Gamma_i$$

$$\psi = \frac{1}{4}\begin{bmatrix} 8 \\ -4 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$\phi_i = \Gamma_i - \psi$ **[differece between each image and average]**

$$\phi_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\phi_4 = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

Covariance matrix $C = \frac{1}{M}AA^T$

$$A = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & -2 & 2 & 0 \\ -3 & 0 & -1 & 4 \\ 1 & -1 & 3 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & 0 & -3 & 1 \\ -1 & -2 & 0 & -1 \\ 1 & 2 & -1 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}$$

For ease we calculate A^TA**, matrix size (MxM)**

Consider eigenvectors v_i **of** A^TA**,**

$(A^TA)v_i = \mu_i v_i$

Premultiply A on both sides,

$AA^TAv_i = \mu_i Av_i$

$(A^TA)u_i = \lambda_i u_i$

Where $Av_i = u_i$ **and** $\lambda_i = \mu_i$

A^TA **and** AA^T **have the same eigenvalues.**

Compute eigenvalues and eigenvectors of C from the covariance matrix C

$$AA^T = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & -2 & 2 & 0 \\ -3 & 0 & -1 & 4 \\ 1 & -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -3 & 1 \\ -1 & -2 & 0 & -1 \\ 1 & 2 & -1 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 4 & 4 & 8 & 4 \\ 4 & 8 & -2 & 8 \\ 2 & -2 & 26 & -2 \\ 4 & 8 & -2 & 12 \end{bmatrix}$$

$$C = \frac{1}{M} AA^T = \frac{1}{4} \begin{bmatrix} 4 & 4 & 8 & 4 \\ 4 & 8 & -2 & 8 \\ 2 & -2 & 26 & -2 \\ 4 & 8 & -2 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -\frac{1}{2} & 2 \\ \frac{1}{2} & -\frac{1}{2} & \frac{13}{2} & -\frac{1}{2} \\ 1 & 2 & -\frac{1}{2} & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -0.5 & 2 \\ 0.5 & -0.5 & 6.5 & -0.5 \\ 1 & 2 & -0.5 & 3 \end{bmatrix}$$

Now we find the eigenvalues and corresponding eigenvectors of covariance matrix

$$\lambda_1 = 5.05, v_1 = \begin{bmatrix} 0.642 \\ 0.802 \\ 0.399 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 0.16, v_2 = \begin{bmatrix} 7.22 \\ -5.26 \\ -0.905 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 6.74, v_3 = \begin{bmatrix} -3.62 \\ 0.852 \\ -11.3 \\ 1 \end{bmatrix}$$

$$\lambda_4 = 5.05, v_4 = \begin{bmatrix} -0.783 \\ -0.812 \\ 0.0816 \\ 1 \end{bmatrix}$$

Now we normalize all the eigenvectors. That is we make, $||u_i|| = 1$ **We compute** $||v_1||, ||v_2||, ||v_3||$**and**$||v_4||$ **and divide the eigenvectors by their length.**

$||v_1|| = \sqrt{0.14 + 0.64 + 0.16 + 1} = \sqrt{2.21}$

$||v_2|| = \sqrt{52.13 + 27.67 + 2.82 + 1} = \sqrt{81.62}$

$||v_3|| = \sqrt{13.104 + 0.72 + 127.7 + 1} = \sqrt{142.52}$

$||v_4|| = \sqrt{0.61 + 0.66 + 0.01 + 1} = \sqrt{2.28}$

Normalized eigenvectors (eigenfaces)

$$u_1 = \frac{1}{\sqrt{2.21}} \begin{bmatrix} 0.64 \\ 0.802 \\ 0.399 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{81.62}} \begin{bmatrix} 7.22 \\ -5.26 \\ -0.905 \\ 1 \end{bmatrix}$$

$$u_3 = \frac{1}{\sqrt{142.52}} \begin{bmatrix} -3.62 \\ 0.852 \\ -11.3 \\ 1 \end{bmatrix}$$

$$u_4 = \frac{1}{\sqrt{2.28}} \begin{bmatrix} -0.783 \\ -0.812 \\ 0.0816 \\ 1 \end{bmatrix}$$

To reconstuct the image I_i

$$\Gamma_i = \psi + \sum_{i=1}^{m} u_i W_i$$

$m = k$ **if we select** k **most important eigenvector, for** k **largest eigenvalues. Weight** (W_k) **is the product of image (vector) with each of the eigenvectors**

$W_k = u_k^T \phi_i$

where $\phi_i = \Gamma_i - \psi$

$\Gamma_1 = \psi + [u_1 W_1 + u_2 W_2 + u_3 W_3 + u_4 W_4]$ **where the weights of image** I_i **are**

$$W_1 = u_1^T \phi_1 = \frac{1}{\sqrt{2.21}} \begin{bmatrix} 0.64 & 0.802 & 0.399 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2.21}} 0.837 = -0.563$$

$$W_2 = u_2^T \phi_1 = \frac{1}{\sqrt{81.62}} \begin{bmatrix} 7.22 & -5.26 & -0.905 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{81.62}} \times -0.837 = -0.563$$

$$W_3 = u_3^T \phi_1 = \frac{1}{\sqrt{142.52}} \begin{bmatrix} -3.62 & 0.852 & -11.3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} = 3.226$$

$$W_4 = u_4^T \phi_1 = \frac{1}{\sqrt{2.28}} \begin{bmatrix} -0.783 & -0.812 & 0.0816 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} = 1.02$$

Weights for image I_1

$$\Omega_1 = \begin{bmatrix} -0.563 & -0.388 & 3.226 & 1.02 \end{bmatrix}$$

Similarly, we can calculate the weights for other images in the training set I_2, I_3 and I_4

Reconstructed image I_i

$\Gamma_1^r = \psi + [u_1W_1 + u_2W_2 + u_3W_3 + u_4W_4]$

$$u_1W_1 = \frac{1}{\sqrt{2.21}} \begin{bmatrix} 0.64 \\ 0.802 \\ 0.399 \\ 1 \end{bmatrix} \quad (-0.563) = \begin{bmatrix} -0.24 \\ -0.303 \\ -0.151 \\ -0.38 \end{bmatrix}$$

$$u_2W_2 == \frac{1}{\sqrt{81.62}} \begin{bmatrix} 7.22 \\ -5.26 \\ -0.905 \\ 1 \end{bmatrix} \quad (-0.38) = \begin{bmatrix} -0.304 \\ 0.221 \\ 0.04 \\ -0.042 \end{bmatrix}$$

$$u_3W_3 = \frac{1}{\sqrt{142.52}} \begin{bmatrix} -3.62 \\ 0.852 \\ -11.3 \\ 1 \end{bmatrix} \quad (3.27) = \begin{bmatrix} -0.99 \\ 0.234 \\ -3.09 \\ 0.28 \end{bmatrix}$$

$$u_4W_4 = \frac{1}{\sqrt{2.28}} \begin{bmatrix} -0.783 \\ -0.812 \\ 0.0816 \\ 1 \end{bmatrix} \quad (1.02) = \begin{bmatrix} -0.53 \\ -0.55 \\ 0.056 \\ 0.675 \end{bmatrix}$$

$$\Gamma_1^r = \psi + \begin{bmatrix} -0.24 \\ -0.303 \\ -0.151 \\ 0.38 \end{bmatrix} + \begin{bmatrix} -0.304 \\ 0.221 \\ 0.04 \\ -0.042 \end{bmatrix} + \begin{bmatrix} -0.99 \\ 0.234 \\ -3.09 \\ 0.28 \end{bmatrix} + \begin{bmatrix} -0.53 \\ -0.55 \\ 0.056 \\ 0.675 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -2.064 \\ -0.398 \\ -3.145 \\ 1.293 \end{bmatrix} = \begin{bmatrix} -0.064 \\ -1.398 \\ -2.145 \\ 0.293 \end{bmatrix}$$

Vector of original image, I_1

$$\Gamma_1^0 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \quad \Gamma_1^r = \begin{bmatrix} 0.999 \\ -1.002 \\ -1.999 \\ -0.001 \end{bmatrix}$$

Similarly

$$\Gamma_2^0 = \begin{bmatrix} 1 \\ -3 \\ 1 \\ -2 \end{bmatrix} \quad \Gamma_2^r = \begin{bmatrix} 1 \\ -2.999 \\ 0.998 \\ -1.999 \end{bmatrix}$$

$$\Gamma_3^0 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \end{bmatrix} \quad \Gamma_3^r = \begin{bmatrix} 3 \\ 0.999 \\ 0.02 \\ -2 \end{bmatrix}$$

$$\Gamma_4^0 = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 0 \end{bmatrix} \quad \Gamma_4^r = \begin{bmatrix} 3.002 \\ -0.998 \\ 4.999 \\ -0.001 \end{bmatrix}$$

$||\Gamma_1^r - \Gamma_1^0|| =$ **Root mean square error to perform recognition of test image**

$$||\Gamma_1^r - \Gamma_1^0|| = \begin{bmatrix} 0.001 \\ 0.002 \\ 0.001 \\ 0.001 \end{bmatrix}$$

$$||\Gamma_2^r - \Gamma_2^0|| = \begin{bmatrix} 0 \\ 0.001 \\ 0.002 \\ 0.001 \end{bmatrix}$$

$$||\Gamma_3^r - \Gamma_3^0|| = \begin{bmatrix} 0 \\ 0.001 \\ 0.002 \\ 0 \end{bmatrix}$$

$$||\Gamma_4^r - \Gamma_4^0|| = \begin{bmatrix} 0.002 \\ 0.002 \\ 0.001 \\ 0.001 \end{bmatrix}$$

Find $W_k^n = u_k^T \phi_{new}$ **where,**$\phi_{new} = \Gamma_{new} - \psi$
$W_1^n = u_1^T \phi_{new}$
$W_2^n = u_2^T \phi_{new}$
$W_3^n = u_3^T \phi_{new}$
$W_4^n = u_4^T \phi_{new}$

$$\Omega_{new} = \begin{bmatrix} W_1^n & W_2^n & W_3^n & W_4^n \end{bmatrix}$$

Eucledian distance e_d **between weight matrix**Ω_{new} **and each face class** Ω_k **is given by**

$$||\Omega_{new} - \Omega_k|| = \sum_{i=1}^{m} (W_1^n - W_1^k)^2$$

is calculated.
When minimum e_d **Threshold(T), the new test face is calssified as unknown. Threshold T is usually taken as half the largest distance between any two face images in training set.**
$T = \frac{1}{2} max_{j,k} ||\Omega_j - \Omega_k|| j,\ k = 1, 2, 3....M$
Here in this section we discussed about the application of eigenvalues

and eigenvectors in face recognition. Eigenfaces are eigenvectors of co-variance matrix, representing given image space. It is basically the set of eigenvectors used in computer vision problem for human face recognition. Any new face image can then be represented as a linear combination of these eigenfaces. This makes it easier to match any two given images and thus face recognition process. We have also used an example to demonstrate it. This is one of the best applications of eigenvectors.

Chapter 3

SOLVING n^{th} ORDER HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS USING EIGENVALUES AND EIGENVECTORS

Diagonalisation

The process of converting a square matrix into its diagonal matrix is called diagonalisation. An n x n matrix is diagonalisable if and only if A has n linearly independent eigenvectors. A non diagonalisable square matrix is called Defective. Consider a matrix A and an invertible matrix P, then $P^{-1}AP$ is a diagonal matrix.
i.e., $P^{-1}AP = D$. Now let us see the proccess of diagonalisation with an example
Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

then the characteristic equation is

$\mathbf{det}(A - \lambda I) = 0$

Writing in matrix form, we get,

$$\begin{vmatrix} 2-\lambda & -3 & 0 \\ 2 & -5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

And on solving this, we get the eigen values:

$\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -4$ **and the eigen vectors corresponding to**

$\lambda_1 = 1$ **is**

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_2 = 3$ **is**

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_3 = -4$ **is**

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

We use these eigenvectors as columns for a 3x3 matrix, say P. As this P is invertible, we find its inverse P^{-1}

i.e,

$$P = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{5} & \frac{3}{5} & 0 \end{bmatrix}$$

Now we find the diagonal matrix D using A,P and P^{-1}

i.e., $P^{-1}AP = D$. i.e.,

$$\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{5} & \frac{3}{5} & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Therefore,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

is the diagonal matrix with entries in the main diagonal as the eigen values of matrix A.

WRITING AN n^{th} ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION IN MATRIX FORM

Consider a 2^{nd} order linear differential equation $2y^{''}$ - $5y^{'}$+y = 0 with the conditions $y(3) = 6$, $y^{'}(3) = -1$

Now put $y(t) = x_1(t)$

$\implies x_1^{'} = y^{'} = x_2$

$y^{'}(t) = x_2(t)$

$\implies y^{''} = \frac{-1}{2}y + \frac{5}{2}y^{'} = \frac{-1}{2}x_1 + \frac{5}{2}x_2$

Converting the initial conditions, $y(3) = x_1(3) = 6$, $x_2(3) = y^{'}(3) = -1$

Now we get system of differential equation,

$x_1^{'} = x_2$, $x_2^{'} = \frac{-1}{2}x_1 + \frac{5}{2}x_2$

$x_1(3) = 6$ **and** $x_2(3) = -1$

In matrix form,

$$\begin{pmatrix} x_1^{'} \\ x_2^{'} \end{pmatrix} = \begin{pmatrix} x_2 \\ -1/2x_1 + 5/2x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1/2 & 5/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now define,

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\bar{x'} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

Then,

$$\bar{x'} = \begin{pmatrix} 0 & 1 \\ -1/2 & 5/2 \end{pmatrix} \quad \bar{x}$$

and

$$\bar{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

3.1 SOLVING FIRST ORDER LINEAR ORDINARY HOMOGENEOUS SYSTEM OF DIFFERENTIAL EQUATIONS

Definition:

If

$$F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ . \\ . \\ f_n(t) \end{bmatrix}$$

is an $n \times 1$ matrix with all the entries as real valued functions and if A is an $n \times n$ matrix of real numbers then the equation

$F'(t) - AF(t) = 0$ **or**

$F'(t) = AF(t)$

is called a first order linear ordinary homogeneous system of differential equations. A solution for such a system is particular function F(t) that satisfies the equation for all values of t.

Lemma 1 ***A real-valued continuously differentiable function*** $f(t)$ ***is a solution to the differential equation*** $f'(t) = af(t)$ ***if and only if*** $f(t) = be^{at}$ ***for some real number b. In our first order system*** $F'(t) = AF(t)$ ***, we replace the real number a by a matrix A. The first order system can be solved easily if A is diagonalizable.***

If A is a diagonal matrix, the system $F'(t) = AF(t)$ **can be written as**

$f_1'(t) = a_{11}f_1(t)$

$f_2'(t) = a_{22}f_2(t)$

.

.

$f_n'(t) = a_{nn}f_n(t)$

and each of these differential equations in the system can be solved separately by the above lemma. If A is diagonalizable, then the general solution has the form

$F(t) = [b_1e^{a_{11}t}, b_2e^{a_{22}t}, ...b_ne^{a_{nn}t}]$

For some $b_1, b_2, \ldots .b_n \epsilon R$

Let us consider the case when A is diagonalizable.

Assume A is a diagonalizable nxn matrix with (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots .\lambda_n$ **corresponding to the eigenvectors in the ordered basis** $B = \{v_1, v_2, \ldots .v_n\}$ **for** R^n**.**

If P is the transition matrix from matrix B to standard coordinates with columns $v_1, v_2, \ldots .v_n$ **and D is the diagonal matrix having eigenvalues** $\lambda_1, \lambda_2, ..\lambda_n$ **along its main diagonal, then**

$F'(t) = AF(t)$

$\Leftrightarrow F'(t) = (PP^{-1}APP^{-1})F(t)$

$\Leftrightarrow F'(t) = PDP^{-1}F(t)$

$\Leftrightarrow P^{-1}F'(t) = DP^{-1}F(t)$

Let $P^{-1}F(t) = G(t)$**, we can see that the original system** $F'(t) = AF(t)$ **is equivalent to the system** $G'(t) = DG(t)$**.**

As, D is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots.\lambda_n$, latter system can be solved as follows:

$G(t) = [b_1 e^{\lambda_1 t}, b_2 e^{\lambda_2 t}, ...b_n e^{\lambda_n t}]$

Now, $G(t) = P^{-1}F(t)$

$\implies PG(t) = F(t)$ **or**

$F(t) = PG(t)$

Since the columns of P are the eigenvectors $v_1, v_2, ..., v_n$, we get

$F(t) = b_1 e^{\lambda_1 t} v_1 + b_2 e^{\lambda_2 t} v_2 + ... + b_n e^{\lambda_n t} v_n$

Thus we have proved the following theorem:

Theorem 1 ***Let A be a diaganolisable*** $n \times n$ ***matrix and let*** $(v_1, v_2, ..., v_n)$ ***be an ordered basis for*** R^n ***consisting of eigenvectors for A corresponding to the (not necessarily distinct) eigenvalues*** $\lambda_1, \lambda_2, \ldots.\lambda_n$***. Then the continuously differentiable solutions for the first order system*** $F'(t) = AF(t)$ ***are all functions of the form***

$F(t) = b_1 e^{\lambda_1 t} v_1 + b_2 e^{\lambda_2 t} v_2 + ... + b_n e^{\lambda_n t} v_n$

where $b_1, b_2, ...b_n \epsilon R$

EXAMPLE

Consider an example, let us solve the first order system, $F'(t) = AF(t)$ **where,**

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Proceding with diagonalisation, we find the eigen values and corresponding eigen vectors of A, as

corresponding to$\lambda_1 = 1$,

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

corresponding to$\lambda_2 = 3$**,**

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

corresponding to$\lambda_3 = -4$**,**

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Then according to theorem 1, the continuously differentiable solutions to the 1^{st} **order system** $F^{'}(t) = AF(t)$ **consists precisely of all functions of the form,**

$$F(t) = \begin{bmatrix} f_1(t), f_2(t), f_3(t) \end{bmatrix} = b_1 e^t \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} + b_2 e^{3t} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + b_3 e^{-4t} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3b_1 e^t + b_3 e^{-4t}, b_1 e^t + 2b_3 e^{-4t}, b_2 e^{3t} \end{bmatrix}$$

Note ***In order to use the above theorem to solve a first order system,*** $F'(t) = AF(t)$***, A must be a diagonalisable matrix. If not we can find solutions to the system using analogous process. i.e, if*** $v_1, v_2, ..., v_n$ ***is a linearly independent set of eigenvectors for A corresponding to the eigenvalues*** $\lambda_1, \lambda_2, \ldots . \lambda_n$***, then the functions of the form***
$F(t) = b_1 e^{\lambda_1 t} v_1 + b_2 e^{\lambda_2 t} v_2 + ... + b_k e^{\lambda_k t} v_k$
are the solutions. However, these are not the only solutions for the system. To find all the solutions, we need to use complex eigenvalues and eigenvectors as well as generalised eigenvectors.

3.2 SOLVING THE HIGHER ORDER HOMOGENEOUS DIFFERENTIAL EQUATIONS

Consider an n^{th} order homogeneous differential equation of the form

$y^n + a_{n-1}y^{n-1} + ... + a_2y'' + a_1y' + a_0y = 0$

Let us understand the method of solving with an example

Consider the differential equation $y''' + 12y'' - 3y' + 4y = 0$

To find solutions for this equation, we define the functions $f_1(t), f_2(t), f_3(t)$ **as** $f_1 = y, f_2 = y', f_3 = y''$

Then we have the system

$f_1' = y' = f_2$

$f_2' = y'' = f_3$

$f_3 = -4f_1 + 3f_2 - 12f_3$

The first and second equations in the system come directly from the definitions of f_1, f_2 **and** f_3**. The third equation is obtained from the original differential equation by moving all terms except** y''' **to the right side.**

Then the above system can be represented as

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 3 & -12 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

That is as

$F'(t) = AF(t)$ **with**

$$F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 3 & -12 \end{bmatrix}$$

Now we can solve this system by using Theorem1

Similarly, we can generalize this method by many homogeneous higher order differential equations

$y^n + a_{n-1}y^{n-1} + \ldots. + a_1y' + a_0y = 0$

Then this equation can be represented as a linear system

$F'(t) = AF(t)$

Where $F(t) = [f_1(t), f_2(t), \ldots .f_n(t)]$

with

$f_1(t) = y, f_2(t) = y', ...f_n(t) = y^{(n-1)'}$ **and where**

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & . & . & . & -a_{n-1} \end{bmatrix}$$

Then this system can be solved using Theorem 1

Result 1 $y^n + a_{n-1}y^{n-1} + ... + a_1y' + a_0y = 0$ ***is represented as a linear system*** $F'(t) = AF(t)$***, where*** $F(t)$ ***and*** A ***are as just described, then*** $\rho A(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ ***Where*** $\rho(A) = 0$ ***is called the characteristic equation of the original differential equation.***

Result 2 $y^n + a_{n-1}y^{n-1} + ... + a_1y' + a_0y = 0$ ***is represented as a linear system*** $F'(t) = AF(t)$***, where*** $F(t)$ ***and*** A ***are just as described, and if*** λ ***is any eigenvalue for A, then the eigenspace*** E_λ ***is one dimensional and is spanned by the vector*** $[1, \lambda, \lambda^2, ..., \lambda^{n-1}]$

Combining the above facts, we can state the solution set for the n^{th} **order homogeneous differential equation directly.**

Consider the differential equation

$y^n + a_{n-1}y^{n-1} + ... + a_2y'' + a_1y' + a_0y = 0$

Suppose that $\lambda_1, ..., \lambda_n$ **are** n **distinct solutions to the characteristic equation**

$x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0$

Then all continuously differentiable solutions of the differential equa-

tion have the form

$y = b_1e^{\lambda_1 t} + b_2e^{\lambda_2 t} + ... + b_ne^{\lambda_n t}$

3.3 APPLICATIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Until now, we've seen how to solve n^{th} order homogeneous ordinary linear differential equations and the importance of eigenvalues and eigenvectors in that process.
Linear differential equations finds its use in many areas of study like medicine, biology, physics ,chemical engineering , economics and engineering fields like to recognise the growth of certain diseases in the human body, describing the motion of waves or pendulum like Simple Harmonic Motion, to find out better investment strategies to assist economists , in the prediction of motion of electricity , and so on.

In this section we explain a few of the prominent applications of linear ordinary differential equations which are : Application in analysis of population growth, Newton's second law of Motion, and Determining the height of a falling object.

3.3.1 POPULATION GROWTH

The importance of linear differential equations in population growth can be understood by considering a mathematical model which governs the population dynamics of a certain species. Earliest attempts of mathematical modelling of human population growth was by the English economist Thomas Malthus in 1798.
Population Growth rate is a quantity that shows the change in population size as a factor of time.
The standard formula for calculating growth rate is

$G_r = \frac{N}{t}$

Where, G_r- Growth rate expressed as a number of individuals
N- Total change in population size for the entire time period

t- Time usually expressed in number of years.

Now, according to Malthusian model, rate of change of population growth of a country at a certain time period is proportional to the total population of the country at that time. That is, if $P(t)$ denoted the total population at time t, then the rate of change of population size,

$\frac{dP(t)}{dt} = kP(t)$

Where k is called the growth constant or decay constantas per the change.

The solution to this linear differential equation will provide population at any future time t.

If $k > 0$, there is a growthin the population and if $k < 0$, there is a decay.

The solution to the linear differential equation is

$P(t) = P_0 e^{kt}$

Where $P(0) = P_0$, is the initial population.

This mathematical model can be used to find the population growth and decay of insects, animals and humans at any time and place

3.3.2 NEWTON'S SECOND LAW OF MOTION

Newton's second law of motion states that, the rate of change of momentum of an object is proportional to the applied unbalanced force in the direction of force.

i.e, $F = ma$

or $F = \frac{dp}{dt}$

where p is the momentum (impulse) of the body.

Now, we know that $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

$\Rightarrow \frac{dp}{dt} = m\frac{d^2x}{dt^2}$

or $F = m\frac{d^2x}{dt^2} \rightarrow (1)$

Where x is the distance of a point mass at any instant t from origin

Equation (1) is a 2^{nd} order differentail equation for the functions $x(t)$.

$x(t)$ is a function whose 2^{nd} derivative with respect to time is a constant $\frac{F}{m}$.

i.e, we have

$\frac{dx}{dt} = \frac{F}{m}t + c_1 \rightarrow (2)$

where the constant does not depend on time.

As the left hand side represent velocity, right hand side must be fixed by the value of velocity at a particular time, say $t = 0$.

Substituting for $t = 0$ in equation(2), we get

$c = v_0$

i.e, $v(t) = \frac{F}{m}t + v_0$

On integrating this again, we have

$x(t) = \frac{1}{2}\frac{F}{m}t^2 + v_0 + c_2$

The constant c_2 is fixed by the value of position at another particular time, say $t = 0$ and the position is x_0. Then the final solution becomes

$x(t) = \frac{F}{2m}t^2 + v_0 t + x_0$

If we know the values of initial position and velocity, the above equation helps us to find the position of the object at any time.

This is how Newton's second law of motion gives rise to a 2^{nd} order linear differential equation.

3.3.3 APPLICATION OF DIFFERENTIAL EQUATION IN FALLING OBJECT

Let an object be dropped from a height h at a time $t = 0$. If $h(t)$ is the height of the object at time t, $a(t)$ is the acceleration and $v(t)$ is the velocity, then

$a(t) = \frac{dv}{dt}$ **and** $v(t) = \frac{dh}{dt}$

For a falling object, $a(t)$ is constant and is equal to $g = -9.8m/s^2$. From the above equation

$\frac{dh}{dt} = v(t)$

$\Rightarrow \frac{d^2h}{dt^2} = \frac{dv(t)}{dt} = \frac{dv}{dt} = a(t) = g$

$\Rightarrow \frac{d^2h}{dt^2} = g \rightarrow (1)$ **Integrating both sided of the above equation, we get**

$\frac{dh}{dt} = gt + v_0$

Where v_0 is the constant velocity at time $t = 0$

Integrating again, we get

$h(t) = \frac{1}{2}gt^2 + v_0 + h_0$

Where h_0 is the initial height.

The above equation which represents the height of a falling object from an initial height h_0 at an initial velocity v_0, as a function of time is a 2^{nd} order linear differential equation.

REFERENCES

[1] Elementary Linear Algebra, Stephen Andrilli and David Hecker, Fourth Edition.

[2] https://1lib.in/book/876294/aab849 Linear algebra and its applications, 4th edition 2005- Gilbert Strang

[3] https://medium.com/swlh/eigenvectors-and-eigenvalues-face-recognition-eigen-faces-f82ebb0b1973

[4] International Journal of Engineering Development and research(IJEDR 2018, Volume 6, issue 2) http://www.ijedr.org/

[5] https://1lib.in/book/876294/aab849 Linear algebra and its applications, 4th edition 2005- Gilbert Strang

[6] http://eng.libretexts.org/

[7] https://youtu.be/mAMTXJJQBDI

[8] A ROBUST TECHNIQUE IMPLEMENTATION FOR FACIAL RECOGNITION UNDER EIGEN FEATURE EXTRACTION, Journal of Global Research in Computer Science, Volume 4, No. 8, August 2013

[9] Shrldon Axler, Linear Algebra Done Right, third edition, Springer Nature, 2014.

[10] Peter Woolf, Engineering LibreTexts.

[11] Deveendra Kumar, Linear algebra concepts and applications, first edition, Alpha Science International Ltd, 2012.

[12] A Human Facial Expression Recognition Model based on Eigen Face Approach Anurag Dea,Ashim Sahaa, Dr. M.C Palb - International Conference on Advanced Computing Technologies and Applications (ICACTA- 2015)

[13] Engineering Applications of Differential equations B. Sumithra, International Journal of Application or Innovation in Engineering and Management (IJAIEM), Volume 6, Issue 7, July 2017

[14] Linear algebra and it's application, David C Lay, Third edition, 18 July 2002