PROJECT REPORT

ON

STUDY ON THERMODYNAMIC CHARACTERISTICS OF SCHWARZSCHILD BLACK HOLE AND KERR BLACK HOLE

SUBMITTED BY

SWEETY PEREIRA

REGISTER NO. : AM21PHY014

in partial fulfillment of

the requirements for award of the postgraduate degree in physics

DEPARTMENT OF PHYSICS AND CENTRE FOR RESEARCH

ST.TERESA'S COLLEGE (AUTONOMOUS) ERNAKULAM

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M.Sc. PHYSICS PROJECT REPORT

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Register No	:	AM21PHY014
Year of Study	:	2021-2023

This is to certify that the project **"STUDY ON THERMODYNAMIC CHARACTERISTICS OF SCHWARZCHILD BLACK HOLE AND KERR BLACK HOLE"** is the work done by Sweety Pereira

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DEPARTMENT OF PHYSICS

ST TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM

THE FEAR OF THE LORD IS THE BEGINNING OF WISDOM

ST. TERESA'S COLLEGE (AUTONOMOUS) ESTD 1925

CERTIFICATE

This is to certify that the project report entitled **"STUDY ON THERMODYNAMIC CHARACTERISTICS OF SCHWARZCHILD BLACK HOLE AND KERR BLACK HOLE"** is the bonafide work done by **Ms. Sweety Pereira**(Reg.No: AM21PHY014) under the guidance of **Dr. SUNSU KURIAN THOTTIL** ,Assistant Professor, Department of Physics and Centre for Research, St. Teresa's College, Ernakulam in partial fulfilment of the award of the Degree of Master of Science in Physics, St. Teresa's College, Ernakulam affiliated to Mahatma Gandhi University, Kottayam.

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I, hereby declare that the project work entitled **"STUDY ON THERMODYNAMIC CHARACTERISTICS OF SCHWARZSCHILD BLACK HOLE AND KERR BLACK HOLE"** is a record of an original work done by me under the guidance of **Dr .SUNSU KURIAN THOTTIL**, Assistant Professor, Department of Physics and center for Research, St .Teresa's College (Autonomous), Ernakulam in the partial fulfilment of the requirements for the award of the Degree of Master of Physics. I further declare that the data included in the project is collected from various sources and are true to the best of my knowledge.

Sweety Pereira

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ABSTRACT

Black holes are the astronomical objects which are the direct result of Einstein's General theory of relativity. Study on the thermodynamic quantities of Blackholes is one of the major ongoing research theme. In order to extract various thermodynamic quantities for our study, we investigated the fundamental thermodynamic laws of blackholes. We have also demonstrated how these numbers change depending on the horizon radius. We examined the properties of Schwarzschild and Kerr blackholes.

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CHAPTER 1

Introduction

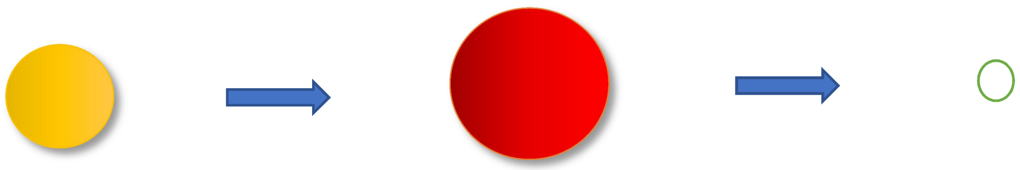
The explanation for the idea of black hole begins when Albert Einstein developed his own theory of relativity known as “General Theory of Relativity” in the year 1915. Actually, the idea of black hole was introduced earlier in 17’s by John Michel, he presented a paper to the Royal Society of London suggesting that when a star got big enough that it is five hundred times wider than our sun, then the gravitational pull will be so strong that the light emitting from such a body is made to return towards it by its own proper gravity. Fifteen years later a French Mathematician Pierre-Simon de Laplace arrived at the same conclusion. But it was the general relativity that explained a genuine Black Hole. General relativity was based on a coincidence that Einstein was struck by, that for an inertial mass of a body that its resistance to a change in its motion is identical to gravitational mass (its change in motion due to gravity force), then he developed ‘Special Theory of Relativity’ which formalises the structure of space and time, which states that space and time are relative and all motion must be relative to a frame of reference. General Relativity brings out the concept of curved space and time. Even though General Relativity was based on the Newtonian Laws, both theories have different concept. According to Newton, gravity is force acting between masses, space was infinite and flat and time was infinite and smooth and never changed. But according to Einstein, space and time exist absolutely, but are not independent, they are interwoven into a single fabric called space-time, in 2-D analogy, we can imagine it like a surface of sheet being bent or distorted, space-time around the Earth by gravity of each massive rotating bodies, like the Earth warp and twist their local space-time. Something different happens when earth is spinning, which leads to a second effect which is the twisting of the space-time contours-‘like a whirlpool’ .Then comes physicalizing of the theories into black holes.

“Any matter could become a Black hole, if you could crush it beyond its Schwarzschild radius.”

If we could crush the sun into the size of a small town, earth into the size of a peanut or a rock into the size of a proton , everything will resembles a black hole. It would have an escape velocity that was the speed of light and nothing could escape, because light is the fastest thing ever exists.

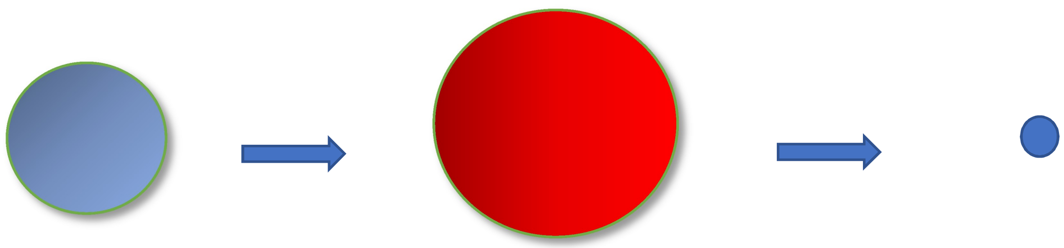
Black hole is the kind of object left behind when a massive star dies. But not in the case of sun, because when a sun dies, it loses its outer envelope, then its 2/3 of mass that remains will crunch down to form a white dwarf, which is a cooling, carbon-rich ember, which is incredibly dense, millions of time denser than the sun now, but formally not the density of a black hole. But a star that started its life 10 times the mass of the sun will lose some mass along the way and then its core, when all fusion steps, there's no energy from fusion gravity will win, and so the inexorable victory of gravity in a massive star in theory leads to a black hole because there is no force to resist compression to that dense state.

The theory or calculations about a black hole is made from Einstein's theory on general relativity. The actual calculations emerged from mostly Robert Oppenheimer and Hans Bethe in the 1930s. And it was essentially a by-product of Oppenheimer's work on the Manhattan Project, figuring out Super dense states of matter, that are how we generated bombs and fusion. He used the same calculations to show that logically a massive star would have no force that could resist it turning into a black hole. Thus death of a massive star forms a black hole.



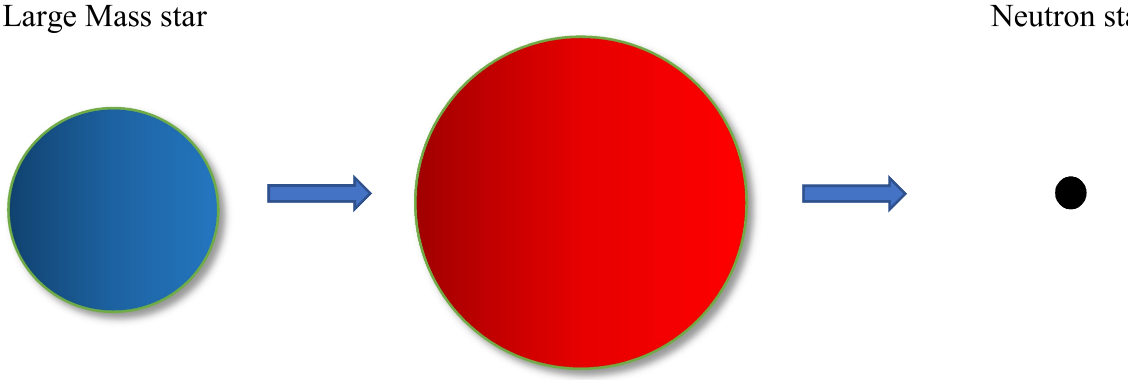
Low to Average Mass star

white Dwarf



Large Mass star

Neutron star



Very Large Mass star

Black Hole

When a massive star exhausts its fuel if the core is more than three times the sun's mass no force can resist the contraction.

It was in 1916, Karl Schwarzschild a German physicist, provide the first exact solution to Einstein's field equations of general relativity which was proposed in 1915. Schwarzschild's solution is identified as a radius for any given mass, known as the Schwarzschild radius, where, if that mass could be compressed to fit within that radius, no known force or degeneracy pressure could stop it from continuing to collapse into a gravitational singularity or black hole. Thus, where the radius of the body is less than its Schwarzschild radius, everything, even photons of light, must inevitably fall into the central body. As a corollary, when the mass density of this central body exceeds a particular limit, it triggers a gravitational collapse to what is known as a Schwarzschild black hole, a non-charged, non-rotating black hole. A general acceptance of the possibility of a black hole did not occur until the second half of the 20th Century, and Schwarzschild himself did not believe in the physical reality of black holes, believing his theoretical solution to be physically meaningless.

In 1958, David Ritz Finkelstein showed that the horizons of black holes are not singular surfaces and determined that whatever falls past the Schwarzschild radius into a black hole cannot escape it. Thus he identified the surface area of Schwarzschild radius.

Black holes in theory are incredibly simpler objects. They are characterized by an event horizon, which is not a physical boundary rather it is an information membrane. It is a boundary between the places of universe we can see and the part that is hidden from universe and that is nothing can escape from and we cannot see inside. It has a particular size that scale linearly varies with the mass of the object. When we look on to a black hole , according to General Theory of Relativity there is a cusp of density that's infinite at the centre and that's called as the Singularity, which is a problem anytime in physics when we get an infinity coming out, we are restricted of getting any information.

In 1963, Roy Kerr solved the gravitational field outside an uncharged rotating massive object, including a rotating black hole and discovered the Kerr geometry.

The other property of a black hole is spin, because the stars that form black holes are spinning, as the collapse, they spin faster and angular momentum gets conserved. So we anticipated all black holes in the universe are spinning and probably very fast. So black holes

are very simple objects with mass, spin and angular momentum. Thus a new theorem which came into existence known as no-hair theorem which states that a Stationary black holes can be completely described by only three parameter, which are mass, charge and angular momentum.
For a long time, people have doubted for the existence of black holes, which was, on the discovery of gravitational waves abolished the doubts on the real existence of black holes. In 2019, the first ever image of black hole and their surroundings which were observed by Event Horizon Telescope were published. In May 2022, astronomers have unveiled the first image of supermassive black hole at the centre of our own Milky Way galaxy.

There were many theories and discoveries which enables studies of black hole alive. The one was of Stephen Hawking's Prediction that black holes have another property beyond the three and the last property is temperature. Hawking saw that there was a very clever mechanism in physics and in lab. It's known that spontaneously from the vacuum, from a pure vacuum of space, particle antiparticle pairs can appear and disappear. That's allowed by Heisenberg's Uncertainty Principle. If that happens near the event horizon of a black hole, there's a finite chance that one member of the pair will be lost inside the black hole, the other will escape. Hawking's contribution about black holes was in a different way. His singular contribution was the prediction that black holes have another property beyond the three and the last property is temperature. Virtual particles and antiparticles pairs are always being created from radiation, then turning into radiation. Hawking realized that one of a pair could pass into the event horizon, so black holes radiate and will eventually evaporate.

And in the aggregate, that is a net loss of either mass and mass and energy, as we know are equivalent. And the radiation from the black hole is called Hawking Radiation. Because this is a very subtle phenomenon, it's a very subtle temperature, Hawking Radiation for a black hole that's a dead star is about a billionth of a kelvin. This is completely immeasurable in astronomy, and may never be measurable unfortunately. Black holes have a phenomenon of slowly losing mass known as Black hole evaporation; this is also a subtle effect. The black holes formed from the death of a star take 10^{68} years to fully evaporate by Hawking Radiation, which cannot be measured by astronomers ever.

These new studies have shown relation between gravity and thermodynamics. Hawking in one of his paper said that the area of event horizon might stay the same or increase with time but it could never decrease always increases when satisfied by energy conditions and never decreased. Thus Bekenstein have clearly observed and said that black holes have entropy and this entropy is proportional to the area of event horizon. Then the four laws of black hole mechanics were given by Bardeen, Carter and Hawking, which had a huge similarity with the four laws of thermodynamics. They refrained from claiming is as the thermodynamic laws of black holes rather it as analogy with the conventional thermodynamics. Hawking thus revealed that the black holes can radiate when quantum effects are taken into consideration. At first hawking had disagreement with the findings of Bekenstein but later he claimed and fixed the proportionality constant of black hole entropy with the horizon area as ¼ in natural unit.

The development of X-ray astronomy provided first evidence for the black hole. With the discovery of gravitational waves, which yet another prediction of Einstein's theory of relativity have proved the existence of black hole. It shows that the black holes don't always exist in isolation they sometimes occur in pairs orbiting around each other. At the time of two black holes orbiting each other, the gravitational interaction between them creates ripples in space-time, which is propagated outward as gravitational waves.

The first observational evidence for black hole was emerged in 1972 , the Cygnus X-1, the binary star system produces the most brightest X-rays of universe since they 33 times the mass of sun. The matter is constantly stripped from the giant star and dragged into an accretion disk around the black hole. Later it was shown that the dark object has a solar mass of 21 which is concentrated into a very small space, which is nothing but a black hole.

Till now we have a lots of indirect evidence for black holes , uncovering all the theories the first direct image of the supermassive black hole at the centre of active galaxy Meisser 87 was observed by Event horizon Telescope and published in April 2019 , which was quite mesmerizing and breath taking.

1.1 General theory of relativity

A generalized framework for the treatment of gravitation and reference frames that are susceptible to gravitational acceleration, the general theory of relativity was developed by physicist Albert Einstein in the early 20th century. It is a geometric explanation of how mass and energy affects the lengths of space-time. It also modifies or improves Newton's original classical theory of gravity. Because it is the weakest of the three forces of nature, the gravitational force is responsible for affecting large-scale phenomena like planets. He generalizes the special theory of relativity and refines newton’s law of universal gravitation which provides a unified description of gravity as a geometric property of space and time or four dimensional space-time.

According to general relativity, mass and energy cause an orthogonal curvature in three spatial dimensions through a fourth. The space-time curvature has an impact on all matter and energy, and gravity results from bodies travelling through this bending of space-time. Based on Einstein's general theory of relativity, cosmology and astrophysics predicted extraordinary natural phenomena including neutron stars, black holes, and gravitational waves. Since moving things must follow the shortest geodesic, light deflection near big objects like stars can reveal the consequences of bent space-time. The light will bend around the distorted space-time brought on by a large star since it is simply taking the quickest route along its course. Karl Schwarzschild, a German physicist, solved Einstein's equations for black holes near a single spherical mass, such as a planet or star, in 1916, not long after Einstein postulated general relativity. Schwarzschild's calculations showed how the curvature of space-time would vary around stars with the same mass but smaller and smaller sizes, or stars that were getting smaller and smaller.

Particularly in regards to the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light, general relativity's predictions diverge dramatically from those of classical physics. Examples of such differences include gravitational redshift, gravitational time dilation and the gravitational time delay. General relativity is the most straightforward explanation that is consistent with experimental facts, however unanswered questions remain. There are significant astrophysical applications for Einstein's theory. It suggests that black holes are real. There is evidence that black holes release powerful radiations. The gravitational lensing phenomena, where several images of the same far-off astronomical object are visible in the sky, are caused by the bending of light by gravity. A direct measurement is what programs like LIGO are aiming for. General relativity also predicts the existence of gravitational waves, which have subsequently been measured indirectly. In addition; general relativity is the basis of current cosmological models of expanding universe.

The alteration of lengths, masses, and time in the presence of a strong gravitational field is a result of general relativity. The addition of simultaneity relativity, kinematic and gravitational time dilation, length contraction, and space-time as a single entity of space and time . The study of elementary particles and their fundamental interactions were altered by relativity, which also brought about the nuclear era in physics.

The two main postulates of the general theory of relativity are:

1. Local physics is governed by general relativity's special theory.
2. The principle of equivalence: There is no way to distinguish between gravity and acceleration.

1.2 Space time metric

A metric used in relativity to define the geometric features of four-dimensional space-time (which relates physical three-dimensional space and time). A space-time metric is defined by an invariant quantity—the square of the four-dimensional interval—that determines the space-time connection (the square of the "distance") between two infinitesimally close events. Space-time diagrams can be used to depict relativistic phenomena, such as why do different observers view where and when events occur differently.

$$ds^2 = \sum_{i=0}^{3}\sum_{k=0}^{3} g_{ik}\, dx^i dx^k \tag{1.1}$$

Where dx^1, dx^2, dx^3 are differences in the spatial coordinates of the events.

$dx^0 = cdt$ (dt - time difference between events, c-speed of light)

g_{ik} are the components of metric tensor.

In general, the metric tensor obeys Einstein's general relativity equations, and the components gn are functions of the coordinates x^1, x^2, x^3, and x^0. The shape of these functions in the chosen frame of reference is determined by the masses existing in space-time. The metric tensor can be simplified to the form in the absence of significant masses.

$$g_{11} = g_{22} = g_{33} = -1, g_{00} = +1, g_{ik} = 0 \tag{1.2}$$

$when\ i \neq k$

For a rectangular Cartesian coordinates;

$$ds^2 = c^2dt - dx^2 - dy^2 - dz^2 \tag{1.3}$$

Such space time metric is a pseudo-Euclidean space (because of – sign preceding $dx^2, dy^2 and\ dz^2$); also called flat space. This metric has the advantage of being Lorentz-invariant, which means that observers in various inertial frames will all measure the same interval ds.

If large masses are present, we can't simplify the metric tensor, which means the space-time is curved and the g_{ik} components will define its curvature. The distribution of masses in space and their motion influence the degree of deviation of a space-time metric from a Euclidean metric.

The space-time interval ds^2 can be positive, negative, or zero, unlike spatial intervals.

- Positive: time-like intervals, events that are causally related.
- Zero: light-like intervals, only something moving faster than the speed of light can link the events
- Negative: space-like intervals, events that are not causally related.

1.3 Einstein's field equations

The equation was published by Albert Einstein in 1915 which is in the form tensor equation that relates the geometry of space-time to the distribution of matter within it. There are ten nonlinear partial differential equations of Einstein field extracted from Albert Einstein's General Theory of Relativity. The EFE describes the basic interaction of gravitation.

The Einstein's field equations are written in the form:

$$G_{\mu\nu} + \Delta g_{\mu\nu} = \kappa T_{\mu\nu} \qquad (1.4)$$

Where $G_{\mu\nu}$ is the Einstein's tensor, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor, Δ is the cosmological constant and κ is the Einstein's gravitational constant.

The Einstein tensor is defined as

$$G_{\mu\nu}=R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \qquad (1.5)$$

Where $R_{\mu\nu}$ is the Ricci curvature tensor and R is the scalar curvature. This is a symmetric second-degree tensor that depends on only the metric tensor and its first and second derivative.

The Einstein's gravitational constant is defined as

$$\kappa = \frac{8\pi G}{c^4} \qquad (1.6)$$

Where G is the Newtonian constant of gravitation and c is the speed of light in vacuum.

The Einstein's field equation can also be written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Delta g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.7}$$

The expression on the left represents the curvature of space-time and on the right is the stress-energy momentum content of space-time. The Einstein's Field Equation can be interpreted as a set of equations dictating how stress-energy momentum determines the curvature of space-time.

1.4 Basic thermodynamics

Thermodynamics is a collection of useful mathematical relations between quantities, every one of which is independently measurable. Though it fails to give microscopic explanation of macroscopic changes it can be used to quantify many unknowns. We can define thermodynamics as a branch of physics which describes the natural processes that includes energy change and laws governing these changes, thus change in temperature plays very important role in it. Thermodynamics deals with some very abstract quantities, and makes deductions using mathematical relations.

Thermodynamics mainly deals with the transformation of heat into mechanical work and vice versa. With the passage of time, the scope of thermodynamics has increased into a vast domain. Thermodynamics deals with heat and its relationship to energy in one of its diverse forms, mechanical, electrical, magnet, chemical or any other. In the kinetic theory all the thermal phenomena are interpreted in terms of disorderly motions of atoms and molecules, but thermodynamics taken no account of atomic constitution of matter. Its deals with the gross characterisation of a system by means of some of its observable properties which are related to its internal state .

Thermodynamics system: thermodynamics is all about the interior of a system, we can specify the system in terms of macroscopic quantities (large scale or bulk scale properties of matter, which have effects on the internal state). Thus, thermodynamic state of a system can be described completely by specifying any two of three quantities: pressure (P), volume (V), and the temperature (θ). These macroscopic quantities are called thermodynamic coordinates or state variables and for a given amount of substance forming the system they are not independent.

The equation connecting these thermodynamics coordinates is called the equation of state and it takes the general form of

$$f(P,V,\theta) = 0 \tag{1.8}$$

If the system consists of homogeneous mixture of several component systems, the concentrations of the different chemical components combining the mixture will be additional variables necessary to define the state of the composite system.

1.5 Black Holes

Although it had originated from Einstein's general theory of relativity back in 1916, the idea of black holes has played a significant role in astronomy and astrophysics since 1950s. And now, 74 years later, the concept of black holes is important to both elementary particle physics and cosmology. Black holes are natural consequence of General Relativity which had made our understanding about space and time deeper. They can be treated as most powerful analytical tool to study macroscopic and microscopic properties of universe. An elementary definition of a black hole is a region of space-time in which the gravitational potential exceeds the square of the speed of light, c^2.Any astronomical body whose escape velocity exceeds the speed of light must be a black hole. The majority of modern astrophysicists contend that if a large star were to shrink up to its event horizon, it would then instantly collapse to a point mass, a mathematical object with zero volume and zero dimensions, which would be the singularity of the resulting black hole.

The boundary of the black hole is Schwarzschild radius which completely depends on the black hole’s mass, which was found by Karl Schwarzschild, it is the physical radius in which any objects become a black hole. This is the radius of event horizon, and anything that passes the horizon radius travels much faster than light. There is no any reflection from the black region which is the event horizon. The hole or dark point comes from the singularity. This singularity is covered by event horizon.

Schwarzschild radius is given as

$$\mathrm{r} = \frac{2GM}{C^2} \tag{1.9}$$

1.5.1 No – hair theorem

The no-hair theorem states that for stationary black hole solutions in general relativity can be completely characterized by only three independent externally observable classical parameters such as mass, electric charge and angular momentum. The variety of information can be found from these parameters since every information disappears behind the event horizon. It was John Wheeler who said that 'black holes have no hair' from which the name of theorem originates.

1.5.2 Classification of Black holes

Based on the no-hair theorem black holes are classified as

i. The static black hole having neither electric charge nor angular momentum are defined by Schwarzschild solution.
ii. The black hole having electric charge and no angular momentum are defined by Reissner- Nordström solutions.
iii. The rotating black holes which no electric charge by Kerr solution.
iv. The rotating black holes with both electric charge and angular momentum by Kerr-Newman solutions.

i. Schwarzschild solution

1n 1916, Karl Schwarzschild solved Einstein's field equations of general relativity. He brings out the solution for gravitational field in empty space around a non-rotating spherically symmetric black hole. Its line element is defined as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\emptyset^2) \qquad (1.10)$$

It exhibits a singularity at the Schwarzschild radius r= 2M, this is the boundary of no escape.

ii. Reissner - Nordström Solution

In 1916 and 1918, Reissner and Finnish independently solved the Einstein-Maxwell field equations for charged spherically symmetric systems. Since black holes have formed from the collapse of stars they can have angular momentum. The Reissner-Nordström geometry describes the geometry of empty space surrounding the charged black hole. Its line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^4}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^4}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\emptyset^2 \tag{1.11}$$

iii. Kerr Solution

The solution for rotating uncharged axially symmetric black holes was put forward by Roy Kerr in 1963. These equations are highly non-linear, which makes exact solutions very difficult to find. Its line element is given by

$$ds^2 = \frac{\Delta}{\rho^2}(dT - hsin^2\theta d\emptyset)^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2 - \frac{sin^2\theta}{\rho^2}[(R^2 + h^2)d\emptyset - hdT]^2 \tag{1.12}$$

Where $h = \frac{L}{M} = angular\ momentum\ per\ unit\ mass$

$$\Delta = R^2 - 2GMR + h^2$$

$$\rho^2 = R^2 + h^2 cos^2$$

iv. Kerr -Newman Solution

They are natural extension to a charged rotating black holes in which the metric is asymptotically flat and have stationary solution of Einstein's Field equations. Its line element is given

$$ds^2 = \frac{\Delta}{\rho^2}(dT - hsin^2\theta d\emptyset)^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2 - \frac{sin^2\theta}{\rho^2}[(R^2 + h^2)d\emptyset - hdT]^2 \tag{1.13}$$

Where

$$\Delta = R^2 - 2GMR + h^2 + GQ^2$$

$$\rho^2 = R^2 + h^2 cos^2\theta$$

1.6 Thermodynamics of black hole

A black hole is characterized only through its mass, its angular momentum and net charges upon it. Other than these three, no single information can be obtained from a black hole. That is what no hair theorem states. A black hole is a monster devouring another star coming in the vicinity of it. Each time black hole swallows up another star, its area increases represented by the growing radius of photon sphere around the event horizon.

Jacob Bekenstein, the father of black hole thermodynamics, showed the similarity between increasing area of the black hole and increasing entropy according to the second law of Thermodynamics. He postulated that the event horizon of the black hole was the measurement of the entropy of the black hole. More the area of the black hole the more will be the entropy. Any matter falling into the black hole, increase the area of the event horizon, thus increasing the entropy. Hence entropy outside the black hole will certainly diminish but overall entropy of the universe will increase subsequently.

Second law of thermodynamics shows that entropy is a function of temperature. Entropy is inversely proportional to the temperature but this relation must imply that for black holes to have entropy they must possess temperature. And if black holes have temperature, then as a black body, it must emit radiation in some form.

It was in 1971, Stephan Hawking proposed the area theorem, which suggests that the total area of the black hole's event horizons should never shrink; which was parallel to the second laws of thermodynamics that states "the entropy within an object never decrease". In 1974 based on the similarities between these two laws, it was suggested that black holes do emit radiation just like any black body and that depends upon the mass of the black hole. Bigger the mass of the black hole, lower will be the temperature and thus it will emit radiation. A smaller black hole will have more temperature and thus emit radiation at greater speed to achieve equilibrium. This radiation was named after Stephan Hawking as the Hawking's Radiation.

There exists a close relationship between thermodynamical parameters and black hole parameters, like the internal energy E to mass m of the black hole, temperature T to surface gravity κ of the event horizon and the entropy S to the area A of the event horizon.

Based on all these studies, the properties of black holes have been classified into four laws called the laws of Black hole Thermodynamics.

They are:

1. Zeroth law: Its states that a simple, non-rotating black hole has uniform surface gravity at its event horizon.
   This is kind of like saying that such a black hole is at thermal equilibrium.
   If area A of the event horizon plays the role of entropy, then surface gravity κ plays the role of temperature (κdA~TdS).the surface gravity is always constant over the event horizon of a stationary. this law is analogous to zeroth law of thermodynamics which states that the temperature is uniform everywhere in a system in thermal equilibrium.
   The significance of the quantity κ lines in the fact that it determines the e-folding time which controls the rate at which the collapsing star red shifts and approaches equilibrium. For a Schwarzschild $\kappa = (4M)^{-1}$.

2. First law: When a little quantity of mass dm is added to a black hole that is going through a quasi-static process, entropy term in first law is dm=κdA/8πG; expression to the first law as it express mass-energy conservation.

3. Second law: Gravity causes systems to expand rather than contract. In the case of black holes, objects can enter them but cannot exit them, making them larger. Similar to the second law of thermodynamics, the size of a black hole is like the entropy; it keeps growing.

4. Third law: According to the cosmic censorship hypothesis, the surface gravity of the horizon cannot be zero (=0), i.e., it cannot be decreased to zero in a limited number of steps; which is the third law.

Analogous with Thermodynamics

Law	In ordinary thermodynamics	In black hole thermodynamics
Zeroth law	Temperature is uniform in a thermodynamic system in equilibrium	Surface gravity throughout the event horizon is uniform
First law	dE= TdS + work terms	dm=$\frac{\kappa}{8\pi G}dA$+ work terms
Second law	dS$\geq$ 0	$dA \geq 0$
Third law	T=0 Cannot be achieved within a finite number of cycles	κ=0 Cannot be achieved within a finite number of cycles

CHAPTER 2

Schwarzschild Black Hole

The Schwarzschild metric, also referred to as the Schwarzschild solution, is an exact solution to the Einstein field equations that describes the gravitational field outside a spherical mass on assuming that the electric charge, mass, angular momentum, and universal cosmological constant are all equal to zero. It is a part of Einstein's theory of general relativity. For describing slowly rotating astronomical objects, such as numerous stars and planets, including Earth and the Sun, the solution is a suitable approximation. Schwarzschild black hole was formed by Karl Schwarzschild in 1916, and for the first time solved the Einstein's field equation.

He also introduced Schwarzschild radial coordinate, which becomes zero at the Schwarzschild radius. The Schwarzschild solutions appears to have singularities at r=0 and r=r_s. Schwarzschild coordinates are divided into two disconnected paths when the singularity is at r=r_s; the exterior Schwarzschild solution for (r>r_s) and the interior Schwarzschild solution($0 \leq r \leq r_s$). If the Schwarzschild solution taken is valid for all r>0, then it is called as a Schwarzschild black hole, thus it is just a point mass and the surrounding empty sphere of intense gravitation.

A Schwarzschild black hole(a static black hole) that lacks both electric charge and angular momentum. The Schwarzschild metric describes a Schwarzschild black hole, and it can only be identified from other black holes by its mass. The Schwarzschild black hole consists of a spherical barrier known as the event horizon, which is located at the Schwarzschild radius, also known as the radius of a black hole. A Schwarzschild black hole is a Schwarzschild solution that is assumed to be valid for all r > 0. It is a totally a valid solution to Einstein's field equations, despite having quite strange features. The Schwarzschild radial coordinate r becomes time like for r<r_s, while the time coordinate t becomes spacelike. The surface r =r_s defines what is known as the black hole's event horizon. It denotes the point beyond which light cannot escape the gravitational field. Any physical object with a radius R less than or equal to the Schwarzschild radius has gravitationally collapsed and become a black hole.

A Schwarzschild black hole has three distinct characteristics:

• A photon sphere that is 1.5 times the size of the Schwarzschild radius

• An event horizon: basically, the black hole's outer surface; its distance from the singularity is the Schwarzschild radius.

• A point singularity with infinite curvature of space and time.

Schwarzschild metric and radius:

A Schwarzschild black hole is static black hole, which has no electric charge or angular momentum.

For spherically symmetric and static body of radius R and mass M, general metric is represented as

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu} dx^{\mu} dx^{\nu} = U(r)dt^2 - V(r)dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \qquad (2.1)$$

where the metric components are

$$g_{00} = U, \quad g_{11} = -V, \quad g_{22} = -r^2, \quad g_{33} = -r^2 sin^2\theta \qquad (2.2)$$

We have Einstein’s Field equation as,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \qquad (2.3)$$

As for Schwarzschild black hole it assumed that the cosmological constant $\Lambda = 0$ and also the stress-energy tensor vanish.

Then, the Einstein’s field equation becomes,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \qquad (2.4)$$

Substituting for $V = \frac{1}{1-\frac{C}{r}}$ $U = \left(1 - \frac{C}{r}\right)$ *and* $C = \frac{2GM}{c^2}$

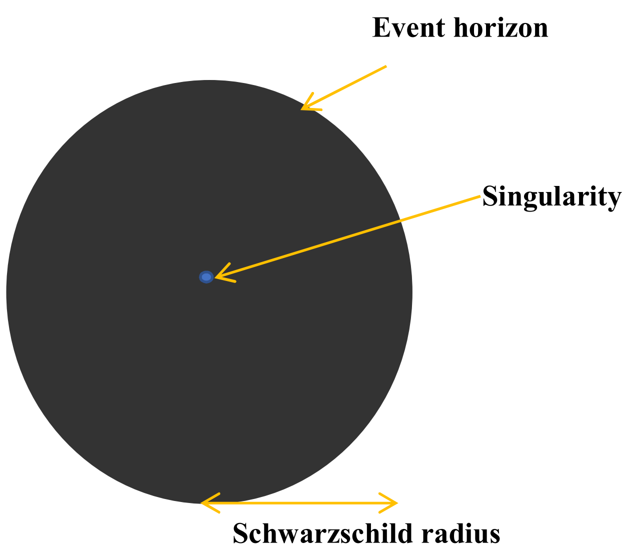
We get the Schwarzschild metric as:

$$\mathbf{ds^2 = (1 - \frac{2GM}{c^2 r})c^2 dt^2 - \frac{1}{(1 - \frac{2GM}{c^2 r})} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2} \tag{2.5}$$

And the Schwarzschild radius is given as:

$$\boldsymbol{r_s = \frac{2GM}{c^2}} \tag{2.6}$$

Where G-gravitational constant, c-speed of light, M-mass of the body.



CHAPTER 3

Gibb's Free Energy Of Schwarzschild Black Hole

In this chapter we tried to find the Gibb's free energy of a Schwarzschild Black hole. By calculating the Gibbs free energies, we can analyze the ranges of horizon radii in which the black holes remain globally stable or prefer the radiation phase.

The generalized form for the entropy of an evaporating Schwarzschild black hole in non-commutative space is given as

$$S \cong \frac{A}{4} - \frac{\pi\alpha}{2} \ln \frac{A}{4} + \sum_{n=1}^{\infty} C_n (\frac{A}{4})^n + C \tag{3.1}$$

Where,

$$C \cong \frac{A_p}{4} + \frac{\pi\alpha}{2} \ln \frac{A_p}{4} - \sum_{n=1}^{\infty} C_n (\frac{4}{A_p})^n \tag{3.2}$$

In the case of commutative space, $\alpha = 0$ and this equation yields the Standard Bekenstein entropy

$$S \cong \frac{A}{4} \tag{3.3}$$

For the spherically symmetric and stationary or Schwarzschild black hole, its surface area is naturally given by the following equation.

$$A = 4\pi R_{bh}^2 \tag{3.4}$$

Where the radius of event horizon for non-spinning and spinning black holes are given by,

$$R_{bh} = \frac{2GM}{c^2} \tag{3.5}$$

And

$$R_{bh} = \frac{GM}{c^2} \tag{3.6}$$

The entropy of black holes(S) can be obtained by putting (4) into equation (3),

$$S = \pi R_{bh}^2 \tag{3.7}$$

The above equation is differentiated as,

$$dS = 2\pi R_{bh} dR_{bh} \tag{3.8}$$

Gibb's free energy G=E-TS

Bekenstein-Hawking formalism of black hole thermodynamics gives the following relation for temperature and entropy of black hole as,

$$T_H = \frac{1}{8\pi M} \tag{3.9}$$

And $$S = 4\pi M^2 \tag{3.10}$$

The product of temperature (T) and entropy (S) is;

$$TS = \frac{1}{8\pi M} 4\pi M^2 \tag{3.11}$$

$$TS = \frac{M}{2} \tag{3.12}$$

According to Einstein well-known mass-energy equivalence relation, we know that $E = Mc^2$

Putting (12) in equation G=E-TS

$$G = Mc^2 - \frac{M}{2} \tag{3.13}$$

Putting c=1 throughout the research work

We have,

$$G = M - \frac{M}{2}$$

$$G = \frac{M}{2} \tag{3.14}$$

The change in free energy of a Schwarzschild black hole due to change in the mass of the black hole can be obtained by differentiating the above equation

$$dG = \frac{1}{2} dM \tag{3.15}$$

First law of black hole thermodynamics gives the relation between change in mass, angular momentum J area A and electrical charge Q,

$$dM = \frac{\kappa dA}{8\pi G} + \Omega\, dJ + \Phi\, dQ$$

For Schwarzschild hole angular momentum and electrical charge is 0;

then

$$dM = \frac{\kappa dA}{8\pi G} \tag{3.16}$$

Using equation (3), we can rewrite equation (16) as,

$$\delta M = \frac{\kappa \delta S}{2\pi} \tag{3.17}$$

Putting equation (8) in (17),

$$\delta M = \kappa R_{bh} \delta R_{bh} \tag{3.18}$$

Substituting δM in equation (15)

$$\delta G = \frac{\kappa}{2} R_{bh} \delta R_{bh} \tag{3.19}$$

This equation gives the change in Gibb's free energy with corresponding change in the event horizon in the terms of surface gravity, mass, angular velocity and event horizon of spinning black holes.

In case of spinning black holes, surface gravity of a black hole is given by the Kerr solution,

$$\kappa = \frac{(M^4 - J_H^2)^{1/2}}{2M\{M^2 + (M^4 - J_H^2)^{\frac{1}{2}}\}} \tag{3.20}$$

Where

$$J_H = a^* \frac{GM^2}{c}$$

Using G=c=h=1,

$$J_H = a^* M^2 \tag{3.21}$$

In the case of spinning black hole having spin parameter ($a^* = 1$),

$$J_H = M^2 \tag{3.22}$$

Using (26) in (24), we have

$$\kappa = 0 \tag{3.23}$$

Equation (23) becomes,

$$\boldsymbol{dG = 0} \tag{3.24}$$

G=constant

From the above derivation it is seen that change in free energy is zero for a Schwarzschild Black hole.

The determination of corrections to the semi-classical BH entropy is an important gradient that should be incorporated in any consistent quantum gravity theory. For a given thermodynamically stable system, thermal fluctuations generally lead to the following corrections for the entropy

CHAPTER 4

Logarithmic Correction To The Entropy Of Schwarzschild Blackhole

Let's begin the discussion by writing metric for spherically symmetric Schwarzschild black hole with singularity at Schwarzschild radius, r = 2M.

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\,\Phi^2) \tag{4.1}$$

This metric is of the form,

$$ds^2 = f(r)dt^2 + \frac{dt^2}{f(r)} + r^2 d\theta^2 + r^2 sin^2\theta d\Phi^2 \tag{4.2}$$

Where metric function is given by,

$$f(r) = 1 - \frac{2M}{r} \tag{4.3}$$

Now, we compute Bekenstein-Hawking temperature

$$T_H = \frac{f'(r)}{4\pi} \quad /(r = r_+) \tag{4.4}$$

$$f'^{(r)} = \frac{2M}{r^2} \tag{4.5}$$

$$T_H = \frac{\frac{2M}{r^2}}{4\pi} = \frac{M}{2\pi r_+^2} \tag{4.6}$$

The value of event horizon radius r_+ can be obtained by setting f(r) =0,

$$f(r) = 1 - \frac{2M}{r} \tag{4.7}$$

$$0 = 1 - \frac{2M}{r}$$

i.e.,

$$r = 2M \tag{4.8}$$

Whenever one goes over from the microscopic to the macroscopic and thermodynamical description of any system, the key role is played by the partition function,

$$\mathrm{Z}(\beta) = \sum_n g(E_n) e^{-\beta E_n} \tag{4.9}$$

Where n is the possible total energies E_n of the system, β is the inverse temperature and g(E_n) is the number of degenerate states of the system associated with the same total energy E_n.

In order to study the effect of thermal perturbations on the entropy of Schwarzschild black hole, we first derive the exact expression for the entropy of Schwarzschild black hole.

In this regard, we write partition function as,

$$Z(\beta) = \int_{0}^{\infty} dE\, \rho(E) e^{-\beta E} \tag{4.10}$$

Where $\beta = \frac{1}{T_H}$ as Boltzmann constant k=1

Now, with the help of inverse Laplace transform, one can get density of states

$$\rho(E) = \frac{1}{2\pi i} \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} d\beta z(\beta) e^{\beta E} \tag{4.11}$$

$$= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} d\beta e^{s(\beta\,)} \tag{4.12}$$

Here s(β)=lnZ(β)+βE represents the exact entropy for the black hole and this depends on temperature explicitly. If one reduces the size of black hole and expands s(β) around equilibrium, then using the method of steepest descent (where $\frac{ds}{d\beta} = 0\; and\; \frac{d^2s}{d\beta^2} > 0)$)we get,

$$S(\beta) = S_0 + \frac{1}{2}(\beta - \beta_0)^2 \frac{d^2s}{d\beta^2} \Big/ + higher\ order\ terms \tag{4.13}$$

Where S_0 represents the equilibrium value of entropy.

By inserting (12) and (13), we have

$$\rho(E) = \frac{e^{s_0}}{2\pi_i} \int d\beta e^{\frac{\frac{1}{2}(\beta - \beta_0)^2 d^2 s}{d\beta^2}} \tag{4.14}$$

On solving integral, we get,

$$\rho(E) = \frac{e^{S_0}}{\sqrt{2\pi \frac{d^2 s}{d\beta^2}}} \tag{4.15}$$

Eventually this leads to,

$$\text{S} = S_0 - \ln(S_0 T_H^2)^{\frac{1}{2}} \tag{4.16}$$

Or

$$\text{S} = S_0 - \frac{1}{2} ln S_0 T_H^2 \tag{4.17}$$

Here, without loss of gravity, we may replace the ½ factor of second term by a more general correction parameter. this is because the coefficient of log term modifies when Hawking temperature has power-law dependence on the entropy of the system. Thus, the corrected entropy by incorporating small fluctuations around thermal equilibrium is given by,

$$\text{S} = S_0 - \alpha \ln(S_0 T_H^2) \tag{4.18}$$

Moreover, 2nd term appears due to small statistical fluctuations around the thermal equilibrium or we can say that it represents the leading-order corrections to entropy of Black hole.

Now, by inserting the value of hawking temperature and Bekenstein entropy to the expression (18), we get perturbed expression for entropy of non-rotating Schwarzschild black hole as,

$$\mathbf{S} = \frac{4\pi G^2 M^2}{c^4} - \alpha \ln \frac{G^2 M^4}{c^4 \pi r_+^4} \tag{4.19}$$

The above equation shows the correction in entropy when thermal fluctuations are incorporated, that is, for a stable thermodynamic system subjected under thermal fluctuations leads to the first order corrections.

We have plotted a graph showing the variations of entropy with horizon radius. (Fig.4.1) and it was found that for increase in the horizon radius there is a corresponding increase in the entropy which satisfies area entropy law of the thermodynamics.

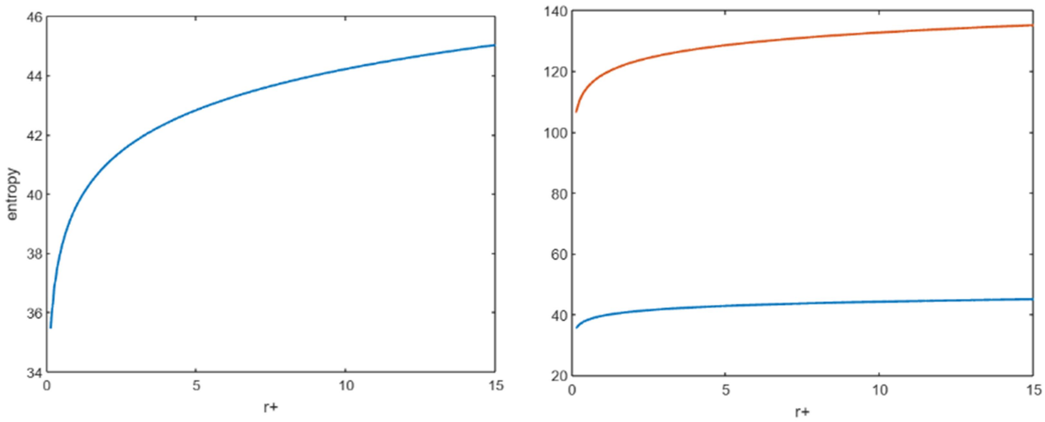


Fig.4.1 Entropy v/s Horizon radius, taking G=M=1 for fluctuations α=1/2 (blue line) and α=3/2(red line)

CHAPTER 5

First Order Corrected Thermodynamic Quantities

In this chapter, we would like to compute thermodynamical quantities by incorporating small fluctuations to the system of Schwarzschild black hole. By knowing the expression of entropy and temperature, we can compute various other thermodynamical quantities.

Helmholtz free energy (F) can be evaluated with the help of the following formula,

$$F = -\int S dT_H \tag{5.1}$$

We know,

$$T_H = \frac{M}{2\pi r_+^2}$$

$$M = T_H 2\pi r_+^2$$

And

$$S = \frac{4\pi G^2 M^2}{c^4} - \alpha \ln \frac{G^2 M^4}{c^4 \pi r_+^2}$$

Substituting M in S and using S in (1),

We get

$$F = -\int \frac{16\pi^3 G^2 T_H^2 r_+^4}{c^4} - \alpha \ln \frac{16\pi^3 G^2 T_H^4 r_+^4}{c^4} . dT_H \tag{5.2}$$

$$\boldsymbol{F = -\frac{16\pi^3 G^2 T_H^3 r_+^4}{3c^4} + \alpha T[\ln(\frac{16\pi^3 G^2 T_H^4 r_+^4}{c^4}) - 4] + c} \tag{5.3}$$

The behavior of Helmholtz free energy with respect to horizon radius can be seen from Fig.5.1

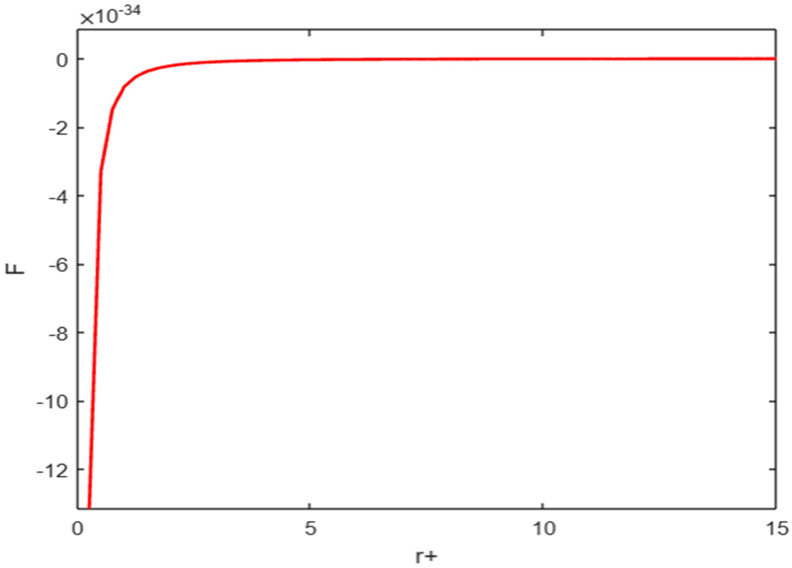


Fig.5.1 Free energy v/s black hole horizon radius taking G=M=1, fluctuations are taken as zero (α=0)

Here we see that the free energy increases as when the horizon radius increases at first, and then it shows a constant behaviour. The plot shows a critical horizon in which free energy further never changes; this point can be interpreted as a phase transition region in which a classical black hole becomes a stable black hole. For larger blackholes (having horizon radius greater than critical horizon radius)have less negative free energy than its equilibrium values, while the smaller ones (having horizon radius less than critical horizon) have more negative free energy values.

We have also studied the free energy v/s horizon radius plot for different α variations, that is incorporating different values of fluctuations.

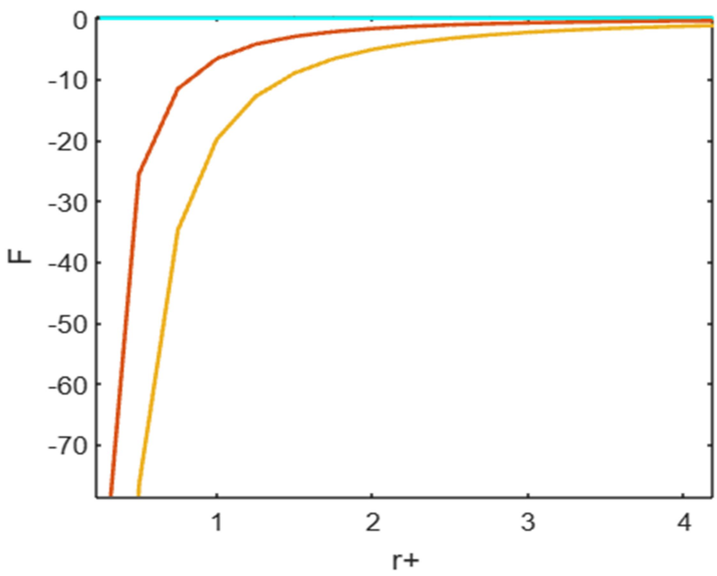


Fig.5.2 Free energy v/s black hole horizon radius taking G=M=1, fluctuations are taken for different α values,α=0(blue line), α=1/2(red line), α=3/2(yellow line)

Here we see that, the Helmholtz free energy shows a decreasing behavior with increasing behavior with horizon. When the α value is taken as zero , we could see that the free energy approaches zero value , when we apply fluctuations the larger blackholes show more variation than smaller ones.

The first law of thermodynamics for uncharged stationary Schwarzschild black hole reads

$$dE = T_H dS \tag{5.4}$$

Which upon integration yields the energy

$$E = \int T_H \, dS \tag{5.5}$$

We know that,

$$T_H = \frac{M}{2\pi r_+^2} \tag{5.6}$$

From S, we obtain

$$M = \frac{c^2}{(4\pi)^{\frac{1}{2}}G}\left\{S^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\left[\ln\left(\frac{G^2M^4}{c^4\pi r_+^4}\right)\right]^{\frac{1}{2}}\right\} \tag{5.7}$$

Substituting M in T_H,

$$T_H = \frac{c^2}{(4\pi)^{\frac{1}{2}}G(2\pi r_+^2)}\left\{S^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\left[\ln\left(\frac{G^2M^4}{c^4\pi r_+^4}\right)\right]^{\frac{1}{2}}\right\} \tag{5.8}$$

(5.9)

$$E = \frac{c^2}{(4\pi)^{\frac{1}{2}}G(2\pi r_+^2)}\left\{S^{\frac{1}{2}} + \alpha^{\frac{1}{2}}\left[\ln\left(\frac{G^2M^4}{c^4\pi r_+^4}\right)\right]^{\frac{1}{2}}\right\}.dS$$

$$\boldsymbol{E = \frac{c^2}{(4\pi)^{\frac{1}{2}}G(2\pi r_+^2)}\left\{\frac{2S^{\frac{3}{2}}}{3} + S\alpha^{\frac{1}{2}}\left[\ln\left(\frac{G^2M^4}{c^4\pi r_+^4}\right)\right]^{\frac{1}{2}}\right\}} \tag{5.10}$$

We have also plotted a graph internal energy v/s horizon radius Fig.5.3

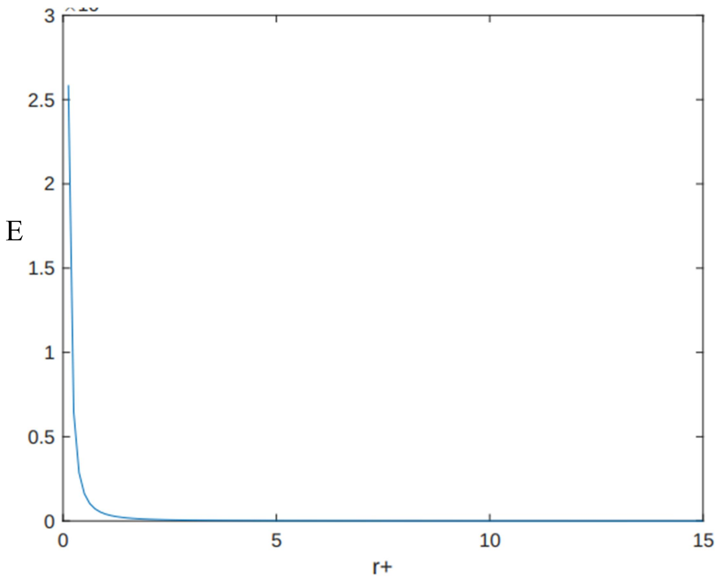


Fig.5.3 Internal energy v/s black hole horizon radius taking G=M=1, fluctuations are taken as zero (α=0)

In the above plot of internal energy at α=0 shows that it decreases as horizon radius increases.

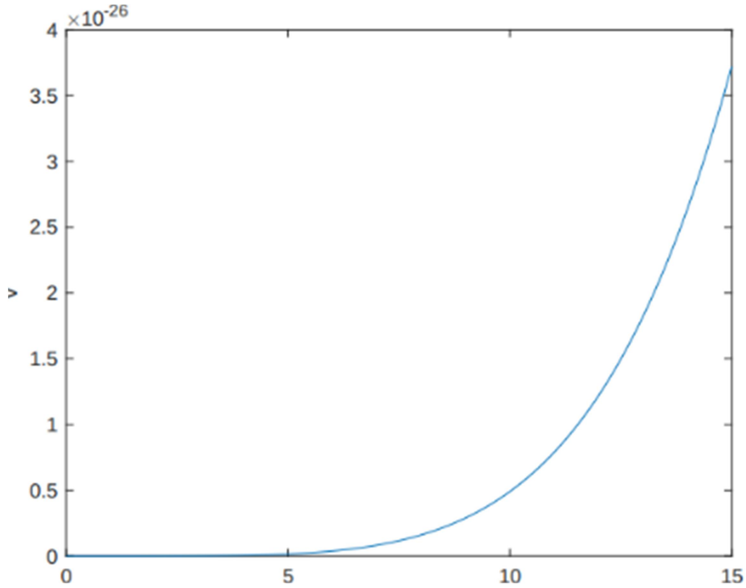
From the area-entropy theorem, we know that the entropy of the black hole is proportional to the area covered by event horizon. So we can find the volume (v) of Schwarzschild black hole,

$$V = 4G \int S_0 dr_+ \tag{5.11}$$

$$V = 4G \int \frac{16\pi^3 G^2 T_H^2 r_+^4}{c^4} dr_+$$

$$\mathbf{V} = \frac{\mathbf{64\pi^3 G^3 T_H^2 r_+^5}}{\mathbf{5c^4}} \tag{5.12}$$

We plot volume v/s horizon radius Fig.5.4 and see that the volume increases as the radius increases for a Schwarzschild blackhole. For the blackholes with small horizon radius the volume is infinitely small that it seems to be very low with respect to the singularity.



` **Fig.5.4Volume v/s black hole horizon radius taking G=M=1**

Since the black holes are considered as thermodynamic systems, we can calculate other macroscopic parameters such as pressure (P), from standard definition of thermodynamics;

$$P = -\frac{dF}{dV} = -\frac{dF}{dr_+}\frac{dr_+}{dV} \tag{5.13}$$

Substituting T_H in (12)

$$V = \frac{16\pi G^3 M^2 r_+}{5c^4} \tag{5.14}$$

And

$$r_+ = \frac{5c^4V}{16\pi G^3M^2} \tag{5.15}$$

Then
$$\frac{dr_+}{dV} = \frac{5c^4}{16\pi G^3M^2} \tag{5.16}$$

Substituting for T_H in $F = -\frac{16\pi^3G^2T_H^3r_+^4}{3c^4}$ and differentiating with respect to r_+,

$$\frac{dF}{dr_+} = \frac{4G^2M^3}{3c^4{r_+}^3} \tag{5.17}$$

Therefore

$$\boldsymbol{P = \frac{-20c^4G^2M^3\pi - 60\alpha Mc^3 + 15\alpha Mc^8\left[\ln\left(\frac{G^2M^4}{\pi c^4r_+^4}\right) + 2\right]}{48\pi^2r_+^3G^3M^3c^4}} \tag{5.18}$$

We plot pressure v/s horizon radius and studied the variations of black hole.

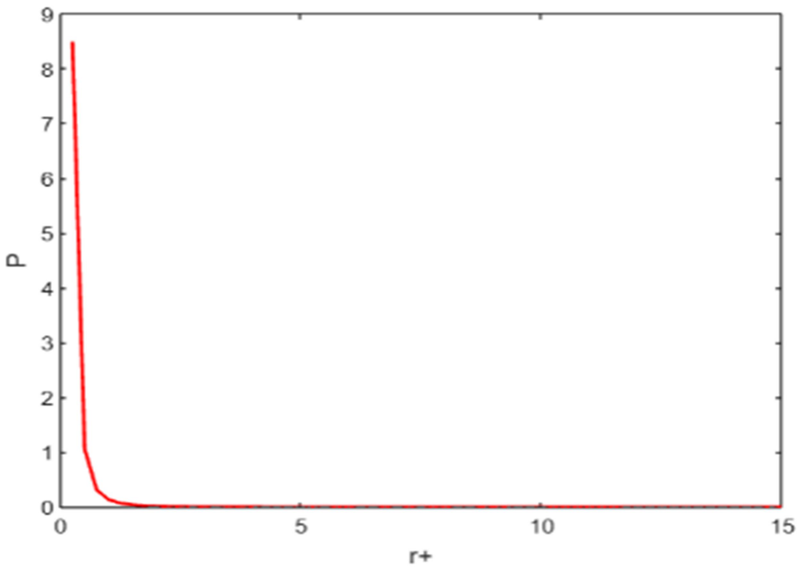


Fig.5.6. Pressure v/s black hole horizon radius taking G=M=1, fluctuations are taken as zero (α=0)

In fig5.7 We observe an intense pressure for small r+ and the pressure decreases as r+ increases.

There is an intense pressure initially while it is shifted to low with increase in the time.

We have also studied the pressure v/s horizon radius plot for different α variations

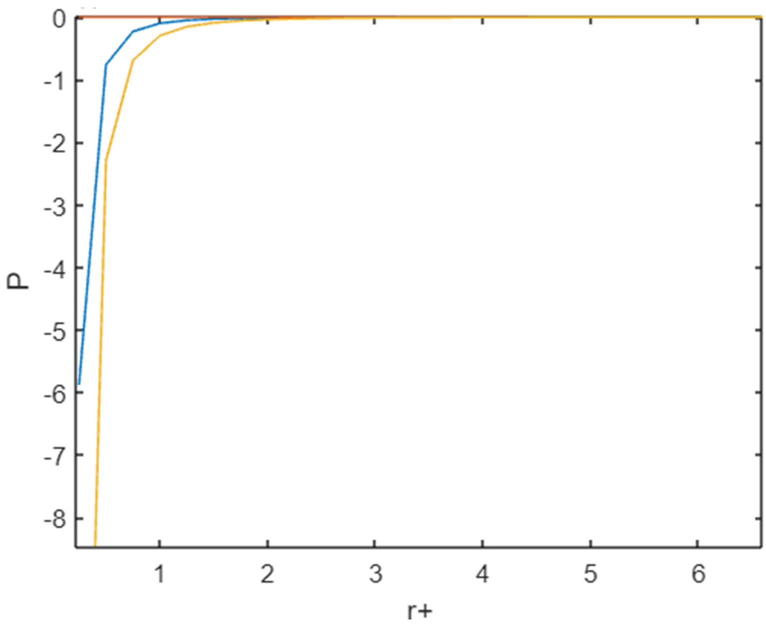


Fig.5.7 Pressure v/s black hole horizon radius taking G=M=1, fluctuations are taken for different α values, α=0(blue line), α=1/2(red line), α=3/2(yellow line)

Here we see that pressure has a negative increase when horizon radius increases which finally approaches zero.

Another thermodynamical quantity enthalpy (H) can be calculated using,

$$H = E + PV \tag{5.19}$$

Using the expression of total energy, pressure and volume, the corrected enthalpy obtained as,

$$\mathbf{H} = \frac{\mathbf{c^6 S^{\frac{3}{2}} - 8\pi^{\frac{3}{2}} G^3 M^3}}{\mathbf{6\pi^{\frac{3}{2}} G r_+^2 c^4}} \tag{5.20}$$

Where

$$S = \frac{4\pi G^2 M^2}{c^4} - \alpha \ln \frac{G^2 M^4}{c^4 \pi r_+^2} \tag{5.21}$$

Variation of enthalpy with respect to black hole radius:

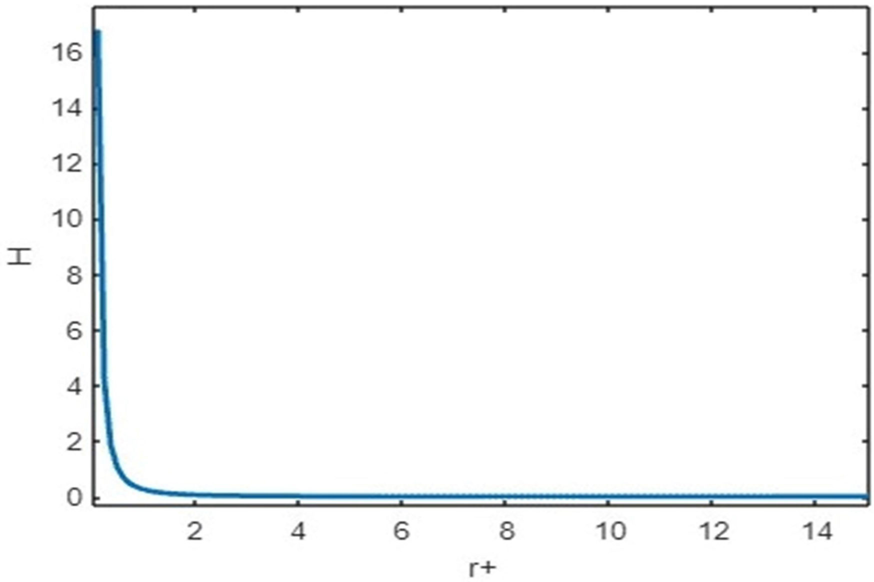


Fig5.8 Enthalpy v/s horizon radius for α=0 taking G=M=1

As the horizon radius increases we see a decreasing behaviour for enthalpy

The Gibb's free energy is the maximum amount of work that can be performed by a thermodynamically closed system at constant temperature and pressure, this can be attained only on reversible process, and the Gibb's free energy for Schwarzschild black hole for small fluctuations can be calculated.

$$G = F + PV \tag{5.22}$$

Using the expression for free energy, pressure and volume the Gibb's free energy (G) is;

$$\mathbf{G} = \frac{-2\mathbf{G}^2\mathbf{M}^3}{3\mathbf{c}^4\mathbf{r}_+^2} - \frac{2\mathbf{c}^4\boldsymbol{\alpha}\mathbf{M}}{\boldsymbol{\pi}\mathbf{c}^4\mathbf{r}_+^2} - \frac{\boldsymbol{\alpha}\mathbf{M}}{2\boldsymbol{\pi}\mathbf{r}_+^2}\ln\left(\frac{\mathbf{G}^2\mathbf{M}^2}{\boldsymbol{\pi}\mathbf{r}_+^4\mathbf{c}^4}\right) + \frac{4}{3\mathbf{c}^4\mathbf{r}_+^2} \tag{5.23}$$

Plot of Gibb's free energy v/s black hole radius:

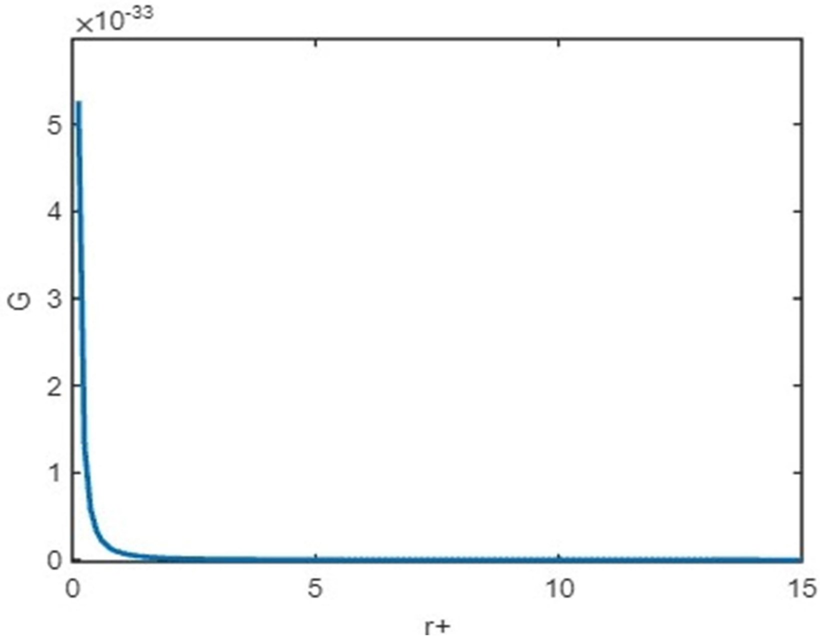


Fig.5.9 Gibb's free energy v/s horizon radius, taking G=M=1 for α=0

From the plot we can see that when there is no fluctuations, as the horizon radius increases it slowly approaches zero. This can be also interpreted as that when the black hole is in a stable state it's Gibb's free energy is nearly zero.

CHAPTER 6
Kerr Black Hole

It was Roy Kerr, who discovered the precise solution to Einstein's equations describing a black hole in 1963. It was later demonstrated that this solution is unique. The Kerr solution describes any spinning (uncharged) black hole and began a revolution in the knowledge of actual black holes from a theoretical perspective. A revolving, neutral black hole is described by the Kerr metric. The only three characteristics that distinguish a black hole are its mass, spin, and electric charge. The geometry of an empty area of space encircling an uncharged, rotating black hole can also be described by the Kerr metric.

Because black holes have angular momentum, the Kerr metric provides some interesting predictions about the space-time surrounding them. For instance, Kerr black holes have two event horizons instead of one, and their shape is more like a squished sphere than a clean sphere. The singularity at the center of a Kerr black hole, however, is a 1-dimensional ring, which is even stranger. The black hole might even be seen if it were spinning quickly enough..

The Kerr metric can expressed in two forms:

1. Boyer-Lindquist form
2. Kerr-Schild form

In Boyer-Lindquist form, in vicinity of mass M and angular momentum J it describes the geometry of space time.

$$ds^2 = \frac{\Delta}{\rho^2}(dT - a\sin^2\theta d\emptyset)^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}[(R^2 + a^2)d\emptyset - adT] \quad (6.1)$$

Where $\frac{J}{Mc} = a$

$\Delta = R^2 - 2GMR + a^2$

$\rho^2 = R^2 + a^2 cos^2\theta$

Also the Kerr solution is represented in kerr-Schild form, which is proposed by Kerr and Schild in 1965 in terms of a particular set of Cartesian coordinates.

$$g_{\mu v} = \eta_{\mu v} + f k_\mu k_v \quad (6.2)$$

$$f = \frac{2GMr^3}{r^4 + a^2 z^2}$$

$$\kappa = (k_x, k_y, k_z)$$

$$k_0 = 1$$

κ is a unit vector, M is the constant mass of spinning object,η is the Minkowski tensor, a is a constant rotational parameter of the spinning object.

The event horizon of the Kerr black hole are found as the radius

$$r_{\pm} = M \pm \sqrt{(M^2 - a^2)}$$

As the event horizon can exists only when $a \leq M$.The "Cosmic censorship" hypothesis states that holes always form with $J \leq M^2$: all singularities of black hole are protected by the horizon.

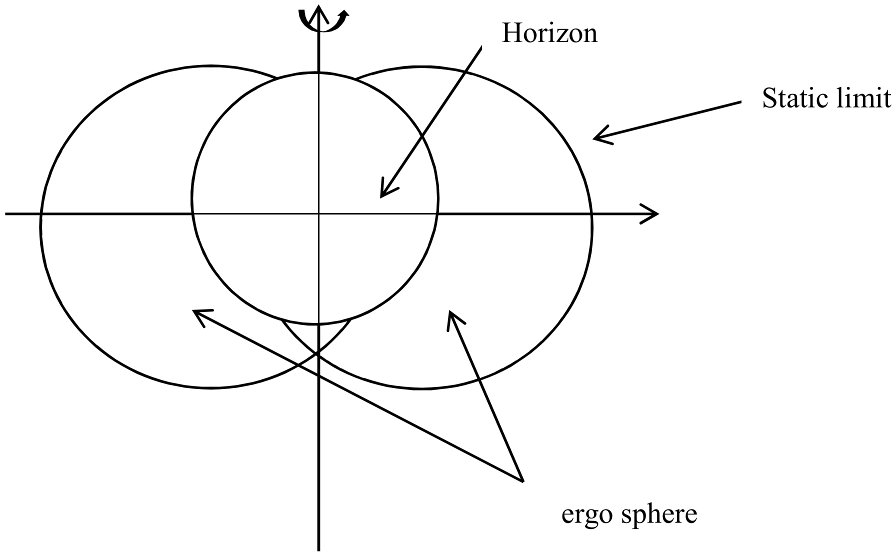
There are two horizons:

1. Outer horizon ($r_{H \equiv} r_+$): it corresponds to Schwarzschild horizon in limit of vanishing a.
2. Inner horizon: also called as Cauchy horizon; where the space-time becomes unstable and is no longer described by the Kerr metric.

There is a place called static limit, where the metric coefficient g_{tt} vanishes:

$$r = M + \sqrt{(M^2 - a^2 cos^2\theta)} \equiv r_{stat}$$

As there is a distance r_{stat} in the black hole, where $g_{tt} = 0$ and beyond which the angular velocity is positive. Therefore in the region $r_{stat} < r < r_H$, there is no static observer: all observers will be dragged along in the direction of rotation of the black hole. This region is known as the ergo sphere.



CHAPTER 7
Logarithmic Correction To The Entropy Of Kerr Black Hole

Let's consider the metric of Kerr black hole to begin our discussion,

$$ds^2 = \frac{\Delta}{\rho^2}(\mathrm{dT} - \mathrm{a}\sin^2\theta\mathrm{d}\emptyset)^2 - \frac{\rho^2}{\Delta}\mathrm{dR}^2 - \rho^2\mathrm{d}\theta^2 - \frac{\sin^2\theta}{\rho^2}[(\mathrm{R}^2 + \mathrm{a}^2)\mathrm{d}\emptyset - \mathrm{adT}] \quad (7.1)$$

Where $\frac{J}{n} = a$

$\Delta=\mathrm{R}^2 - 2GMR + a^2$

$\rho^2 = \mathrm{R}^2+a^2cos^2\theta$

Kerr space-time has coordinate singularities in the axis symmetry ($\theta = 0$) and in those values of r for which ($\Delta= 0$).This last conditions can be written as

$$\Delta= (r - r_+)(r - r_-) \quad (7.2)$$

Then the event horizon of the Kerr black hole are found as the radius,

$$r_\pm = M \pm \sqrt{(M^2 - a^2)} \quad (7.3)$$

Equation (3) makes it possible to distinguish three cases, $M^2 < a^2, M^2 > a^2, M^2 = a^2$.
In addition, it has curvature singularities when, $\rho = 0$.

The angular velocity of the black hole, which is constant at the horizon, can be written as,

$$\Omega = \frac{a}{r_+^2 + a^2} \quad (7.4)$$

Using (3), we have,

$$\Omega = \frac{a}{2M(M + \sqrt{(M^2 - a^2)}} \tag{7.5}$$

Bekenstein proposed that the entropy of a black hole is represented by equation,

$$S = \frac{rk_{\boldsymbol{B}}c^3}{\hbar G}A \tag{7.6}$$

Where r is a dimensionless constant, Hawking found that r corresponds to $Y_{\boldsymbol{4}}$, based as the application of quantum field theory over curved spaces to black holes, then the entropy of black hole can be ,

$$S = \frac{1}{4}\frac{k_{\boldsymbol{B}}c^3}{\hbar G}A \tag{7.7}$$

In the system of natural units ($\hbar = G = c = 1$),

$$S = \frac{A}{4} \tag{7.8}$$

Where $A = 4\pi(r_+^2 + a^2)$ is the area of the event horizon of Kerr black hole,

$$S = \frac{4\pi(r_+^2 + a^2)}{4} = \pi(r_+^2 + a^2) \tag{7.9}$$

Differentiating equation (9) with respect to and assuming that parameter a is small than r and M,

$$dS = 2\pi r_+ . dr \tag{7.10}$$

Using (3),

$$dS = 2\pi \frac{r^2}{\sqrt{(M^2 - a^2)}} dM \tag{7.11}$$

The black hole can be considered as a system in a state of thermodynamic equation that obeys the first law,

$$dM = TdS + \Omega dS + \Psi dQ \tag{7.12}$$

For the case of neutral black hole (dQ=0),

Replacing equation (5) and equation (4) in equation (12),

$$dM = 2\pi T \frac{r^2}{\sqrt{(M^2 - a^2)}} dM + \frac{a}{r_+^2 + a^2} dJ \tag{7.13}$$

By doing a<<M and using J=aM, we find that Hawking temperature is,

$$\boldsymbol{T = \frac{1}{2\pi} \frac{\sqrt{(M^2 - a^2)}}{r_+^2 + a^2}} \tag{7.14}$$

According to equation (3), the hawking temperature for Kerr black hole can finally be represented as,

$$T = \frac{1}{2\pi} \frac{\sqrt{(M^2 - a^2)}}{[2M^2 + 2M\sqrt{(M^2 - a^2)}]} \tag{7.15}$$

The corrected entropy by incorporating small fluctuation around thermal equilibrium is given by,

$$S = S_0 - \alpha \ln(S_0 T_H^2) \tag{7.16}$$

$$S_0 = \pi(r_+^2 + a^2)$$

$$T = \frac{1}{2\pi}\frac{\sqrt{(M^2 - a^2)}}{(r_+^2 + a^2)}$$

Substituting S_0 and T in equation (16),

$$\boldsymbol{S = \pi(r_+^2 + a^2) - \alpha \ln\left(\frac{M^2 - a^2}{4\pi(r_+^2 + a^2)}\right)} \tag{7.17}$$

Where a is the Kerr parameter, $a = 0.9982M$.

We have plotted a graph showing the variations of entropy with horizon radius(Fig.7.1 &Fig.7.2)

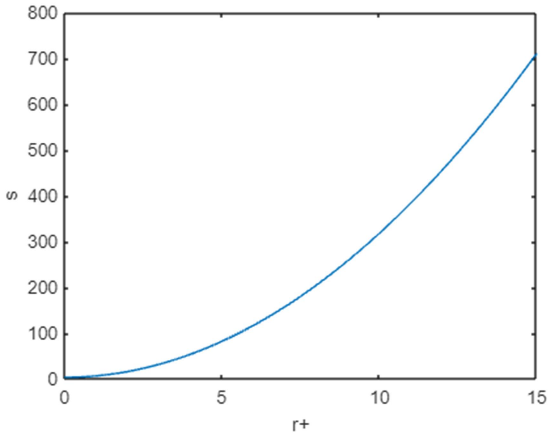


Fig.7.1 Entropy vs Horizon radius taking M=1 for α=0

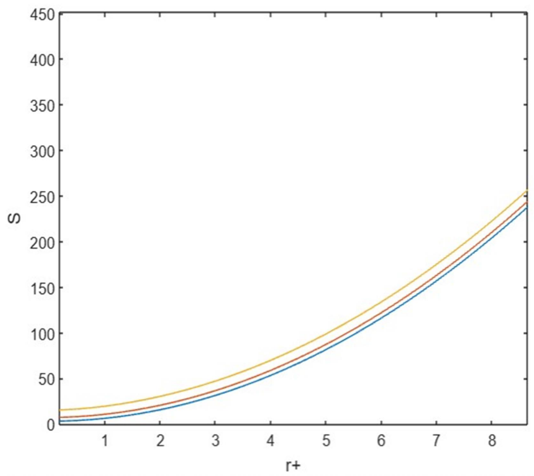


Fig 7.2. Entropy vs Horizon radius taking M=1 for α=0(blue),α=1/2(red),α=3/2(yellow)

From the above plot Fig.7.1& Fig.7.2 w find that the entropy of a kerr blackhole increases with increase in the horizon radius. This satisfies the area - entropy law of thermodynamics.

CHAPTER 8
Thermodynamic Equation of States

By adding minor fluctuations to the systems, we construct various thermodynamic equations of states in this chapter. We begin by calculating Helmholtz free energy. By knowing the equation of entropy and temperature, we calculate free energy,

$$F = -\int S.dT_H \tag{8.1}$$

Substituting in S from T_H,

$$F = -\int S.dT_H = -\int \pi(r_+^2 + a^2) - \alpha\ln(\pi T^2(r_+^2 + a^2)).dT$$

$$\boldsymbol{F = -\pi(r_+^2 + a^2)T + \alpha\{\ln(\pi T^2(r_+^2 + a^2)T - 2T)\} + C} \tag{8.2}$$

We plot a graph showing the variation of Free energy with horizon radius. From the plot we see that Free energy tends to be decreasing in the negative range as we increase the horizon radius.

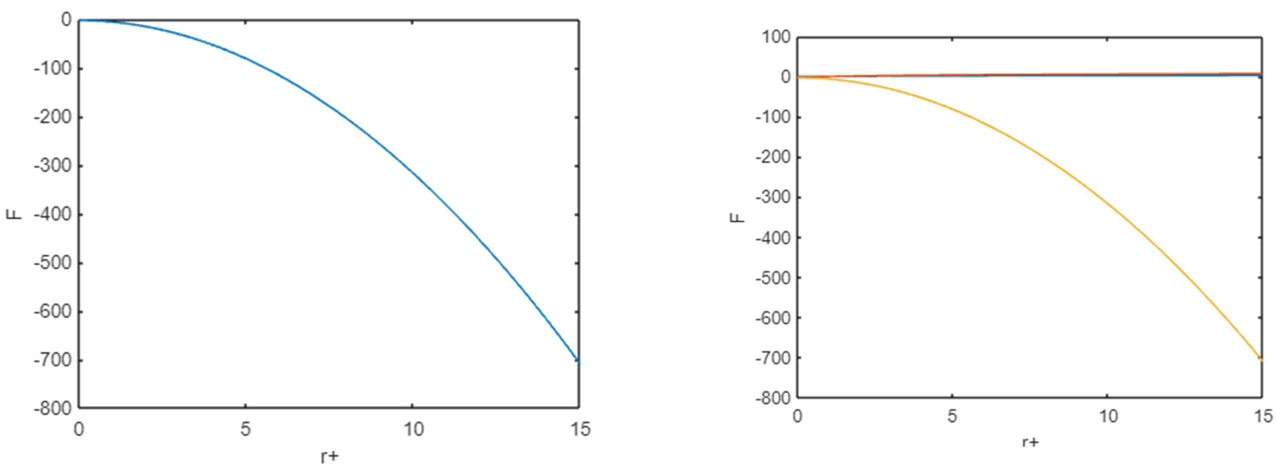


Fig 8.1. Free energy vs Horizon radius taking M=1

The internal energy can be found out by considering the first law of thermodynamics,

$$E = \int T_H dS \tag{8.3}$$

$$S = \pi(r_+^2 + a^2) - \alpha \ln\left(\frac{M^2 - a^2}{4\pi(r_+^2 + a^2)}\right)$$

For $\alpha = 0,\ S = \pi(r_+^2 + a^2)$

Then

$$T = \frac{1}{2\pi}\frac{\sqrt{(M^2 - a^2)}}{(r_+^2 + a^2)} = \frac{\sqrt{(M^2 - a^2)}}{2S} \tag{8.4}$$

Therefore $$E = \int \frac{\sqrt{(M^2 - a^2)}}{2S} dS$$

$$\boldsymbol{E = \frac{\sqrt{(M^2 - a^2)}}{2} ln\left(\pi(r_+^2 + a^2)\right)} \tag{8.5}$$

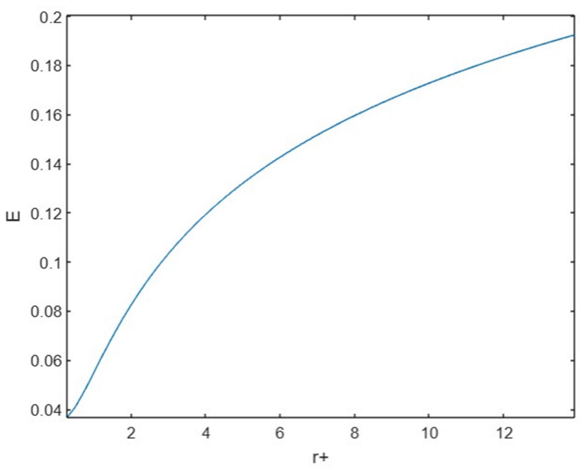


Fig.8.2 Internal energy vs Horizon radius taking M=1

We have plotted internal energy with horizon radius and found that it shows an increasing behavior with increase in the radius.

From the area-entropy theorem, we know that the entropy of the black hole is proportional to the area covered by event horizon. Thus we can find the volume of Kerr black hole.

$$V = 4G \int S_0 dr \tag{8.6}$$

$$V = 4G \int \pi(r_+^2 + a^2).dr$$

$$\boldsymbol{V = \frac{4G\pi r(r^2 + 3a^2)}{3}} \tag{8.7}$$

A graph is plotted showing the variation of volume of Kerr blackhole with the horizon radius. We see that the volume shows an increasing behavior.

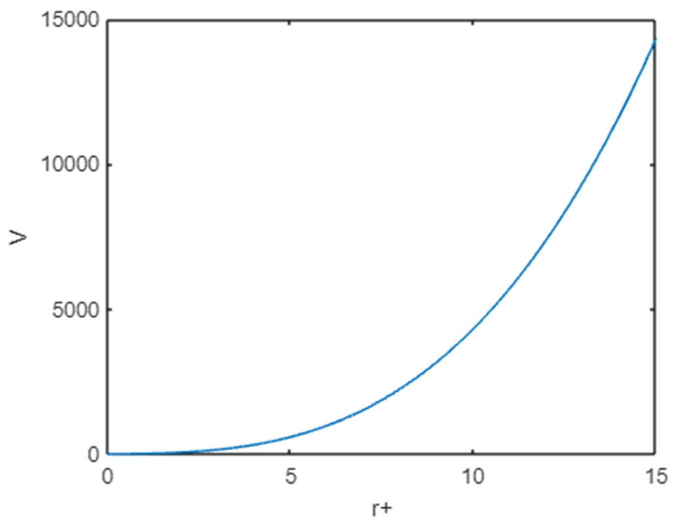


Fig 8.3. Volume vs Horizon radius taking M=1.

As black holes are considered as thermodynamic systems, we can calculate other macroscopic parameters such as pressure (P), from standard definition of thermodynamics

$$P = -\frac{dF}{dV} = -\frac{dF}{dr_+}\frac{dr_+}{dV} \tag{8.8}$$

From V, we have $r = \frac{3V}{4G\pi r(r^2+3a^2)}$

$$\frac{dr_+}{dV} = \frac{d}{dV}\left\{\frac{3V}{4G\pi r(r^2+3a^2)}\right\} = \frac{3}{4G\pi r(r^2+3a^2)} \tag{8.9}$$

And $F = -\pi(r_+^2 + a^2)T$

$$\frac{dF}{dr_+} = \frac{d}{dr_+}\{-\pi(r_+^2 + a^2)T\} = -2\pi r_+ T \tag{8.10}$$

Substituting these we get pressure as,

$$\boldsymbol{P = \frac{3r_+\sqrt{(M^2-a^2)}}{4G\pi(r^2+a^2)(r^2+3a^2)}} \tag{8.11}$$

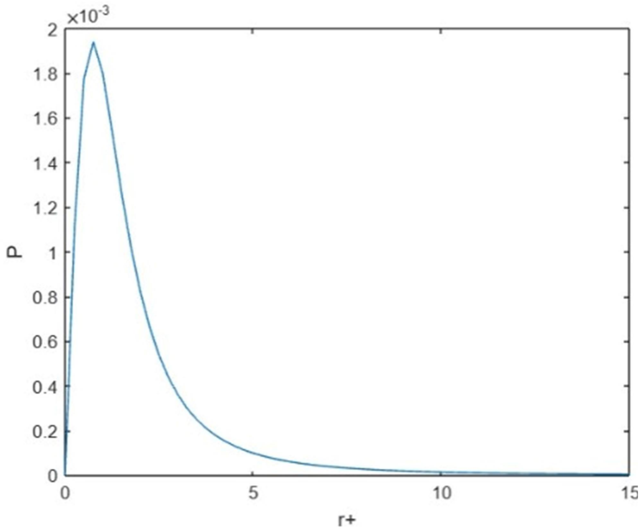


Fig 8.4. pressure vs horizon radius taking M=1,G=1

Plotting pressure with horizon radius shows that though the pressure shows a slight increase initially it suddenly decreases to zero on increasing the horizon radius.

The enthalpy of Kerr black hole can be calculated from

$$H=E+PV \tag{8.12}$$

Substituting for E, P and V

$$\boldsymbol{H = \frac{\sqrt{(M^2-a^2)}}{2}\, ln\left(\pi(r_+^2+a^2)\right) + \frac{r^2\sqrt{(M^2-a^2)}}{(r^2+a^2)}} \tag{8.13}$$

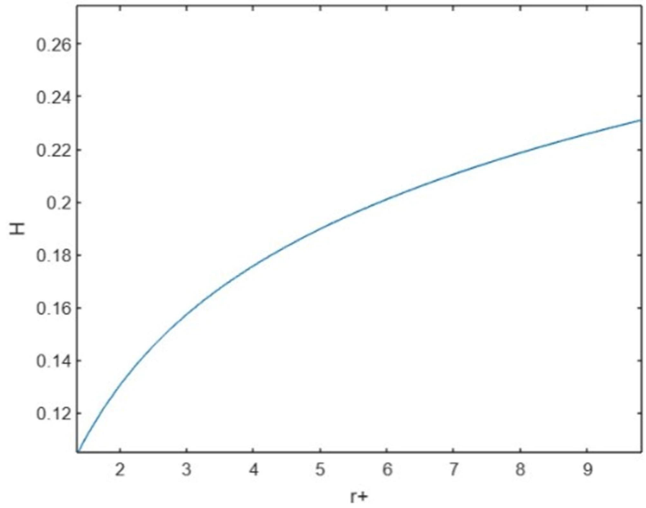


Fig.8.5 Enthalpy vs Horizon radius taking M=1

When we plot enthalpy with horizon radius we see an increasing behavior like the internal energy**.**

Finally the Gibb's free energy of Kerr black hole can be derived by using the equation

$$G=F+PV \tag{8.14}$$

$$\boldsymbol{G=\frac{-\pi(r^2+a^2)^2+r^2\sqrt{(M^2-a^2)}}{(r^2+a^2)}} \tag{8.15}$$

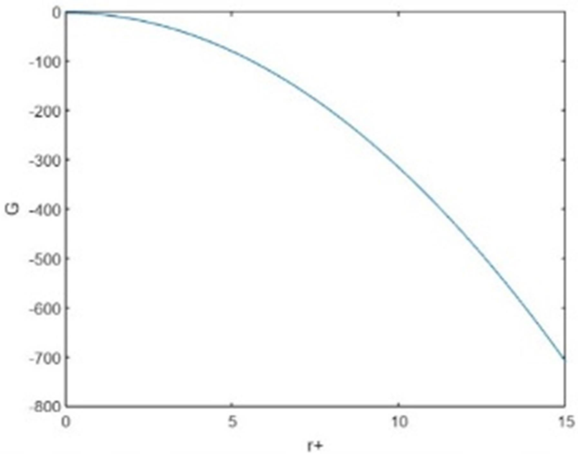


Fig . 8.6 Gibb's free energy vs horizon radius taking M=1

We plot the variation of Gibb's free energy with horizon radius and was seen that as the radius increases Gibb's energy shows a decreasing behavior.

CHAPTER 9

Study on the stability of Black Holes

Numerous investigations on the event horizon, singularities, and the nature of black holes have been conducted since the publication of Einstein's Theory of General Relativity in 1915. Karl Schwarzschild was able to solve Einstein's field equations for a non-spinning, spherically symmetric black hole after that; Roy Kerr later also provided the solution for a rotating black hole. Both the Schwarzschild and Kerr family of black holes are vacuum solutions of the Einstein Equation.

John Wheeler established the foundation for stability studies on the Schwarzschild metric. Understanding stability is crucial since its solutions are quite accurate. We investigate the thermodynamic stability of Schwarzschild and Kerr black holes based on black hole thermodynamics. We compute specific heat capacity by including minor thermal fluctuations, which is one method, and from the positivity of specific heat, we can ensure the local thermal stability of the black hole. This method was developed in general relativity to explore the stability of those black holes. The most necessary conditions P>0, is required for showing that black holes in general relativity to be thermodynamically stable.

Stability of Schwarzschild Black hole

From the most standard relation, we can calculate the specific capacity as,

$$C = \frac{dE}{dT_H} = \frac{dE}{dr_+}\frac{dr_+}{dT_H} \tag{9.1}$$

$$C = \frac{c^2\left[\frac{4\pi}{c^4} - \propto \ln(\frac{1}{c^4\pi r_+^4})\right]^{\frac{3}{2}}}{3 \times 2^{\frac{3}{2}}\pi^2 r_+^3(\frac{M}{2\pi r_+^2})^{\frac{3}{2}}} \tag{9.2}$$

The calculated value of $3 \times 2^{\frac{3}{2}}$ is 8.485 which is approximated as ≈ 8.5, then

$$\boldsymbol{C = \frac{c^2\left[\frac{4\pi}{c^4} - \propto \ln(\frac{1}{c^4\pi r_+^4})\right]^{\frac{3}{2}}}{8.5\pi^2 r_+^3(\frac{M}{2\pi r_+^2})^{\frac{3}{2}}}} \tag{9.3}$$

Specific heat capacity v/s horizon radius for different α values are shown below(Fig:9.1)

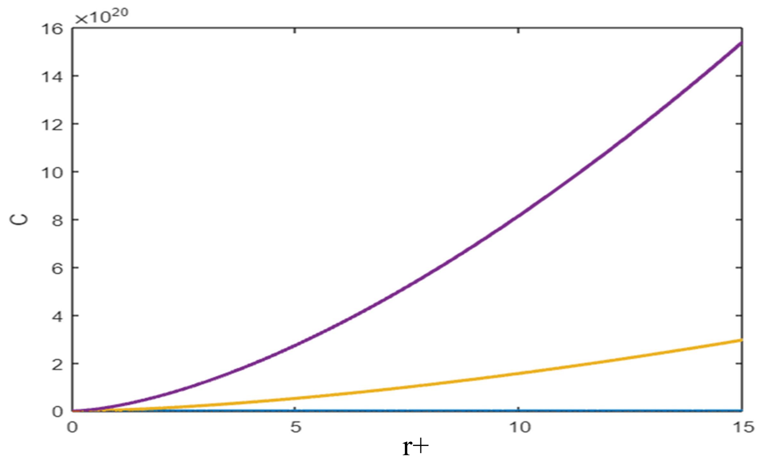


Fig 9. 1.Specific heat capacity v/s horizon radius taking G=M=1 for different fluctuations

From the figure, we see that when the perturbation ∝ is zero, the system shows a linear stability. The linearly curve remains constant on increasing the horizon radius. The equilibrium values of specific heat is positive always indicates that the black hole is in stable phase on the absence of thermal fluctuations. The most general stability seen for black holes are linear stability which assumes that the metric solution is stable when the perturbations are linear. Research work of Dafermos, Holzegel, and Rodnianski etal. would expand their proofs to prove full linear stability for the Schwarzschild black hole. Non-linear stability is also seen, but it doesn't make any assumption on the form perturbations and it can be taken as an open problem now.

Stability of Kerr Black Hole

Specific heat capacity,

$$C = \frac{dE}{dT_H} = \frac{dE}{dr_+}\frac{dr_+}{dT_H} \tag{9.4}$$

$$E = \frac{\sqrt{(M^2 - a^2)}}{2} ln\big(\pi(r_+^2 + a^2)\big) \tag{9.5}$$

$$\frac{dE}{dr_+} = \frac{r\sqrt{(M^2 - a^2)}}{r^2 + a^2} \tag{9.6}$$

And

$$\frac{dr_+}{dT_H} = 2\pi(r^2 + a^2) \tag{9.7}$$

Therefore

$$\boldsymbol{C = 2\pi r\sqrt{(M^2 - a^2)}} \tag{9.8}$$

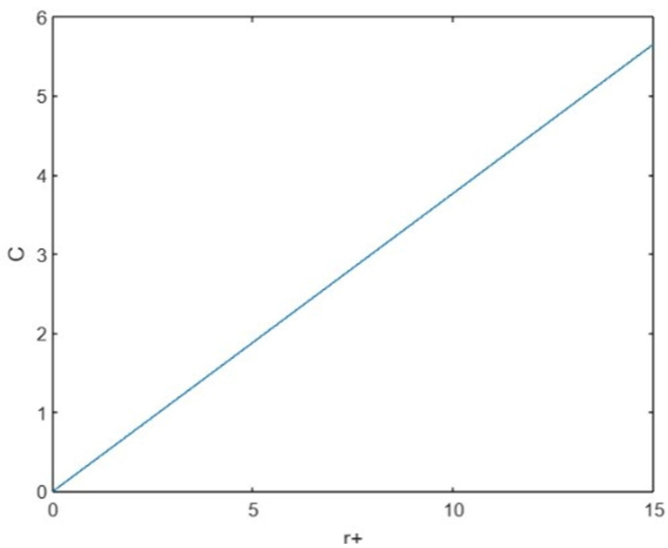


Fig 9. 2:Specific heat capacity vs Horizon radius of Kerr black hole taking M=1

From Fig 9.2, we can see that specific heat capacity increases linearly with increase in the horizon radius. Since it varies linearly, we can see that the blackhole is stable.

Comparison Between the thermodynamic quantities of Schwarzschild and Kerr Black holes

Thermodynamic quantities	**Schwarzschild black hole**	**Kerr black hole**
Metric	Spherically symmetric solution	Stationary axially symmetric solution
Entropy	Increases with Horizon radius $S = \frac{4\pi G^2 M^2}{C^4} - \alpha ln \frac{G^2 M^4}{C^4 \pi r_+^4}$	Increases with Horizon radius $S = \pi(r_+^2 + a^2) - \alpha ln\left(\frac{M^2 - a^2}{4\pi(r_+^2 + a^2)}\right)$
Free energy	At first increases then shows a constant behavior $F = -\frac{16\pi^3 G^2 T_H^3 r_+^4}{3c^4} + \alpha T[\ln(\frac{16\pi^3 G^2 T_H^4 r_+^4}{c^4}) - 4]$	$F = -\pi(r_+^2 + a^2)T + \alpha\{ln(\pi T^2(r_+^2 + a^2)T - 2T)\}$
Internal energy	Decreasing with increase in Horizon radius $E = \frac{c^2}{(4\pi)^{\frac{1}{2}} G(2\pi r_+^2)} \left\{\frac{2S^{\frac{3}{2}}}{3}\right\}$	Increase as the horizon radius increases $E = \frac{\sqrt{(M^2 - a^2)}}{2} ln(\pi(r_+^2 + a^2))$

Volume	Increases(side opening parabola) $V = \frac{64\pi^3 G^3 T_H^2 r_+^5}{5c^4}$	Increases(Top opening parabola) $V = \frac{4G\pi r(r^2 + 3a^2)}{3}$
Pressure	Decreasing behavior $P = \frac{-20c^4 G^2 M^3 \pi + 2}{48\pi^2 r_+^3 G^3 M^3 c^4}$	Decreasing behavior $P = \frac{3r_+\sqrt{(M^2 - a^2)}}{4G\pi(r^2 + a^2)(r^2 + 3a^2)}$
Enthalpy	Decreasing behavior $H = \frac{c^6 S^{\frac{3}{2}} - 8\pi^{\frac{3}{2}} G^3 M^3}{6\pi^{\frac{3}{2}} G r_+^2 c^4}$	Increasing behavior $H = \frac{\sqrt{(M^2 - a^2)}}{2}\, ln(\pi(r_+^2 + a^2)) + \frac{r^2\sqrt{(M^2 - a^2)}}{(r^2 + a^2)}$
Gibb's free energy	As horizon radius increase, it approaches to zero. $G = \frac{-2G^2 M^3}{3c^4 r_+^2} + \frac{4}{3c^4 r_+^2}$	It shows a decreasing behavior. $G = \frac{-\pi(r^2 + a^2)^2 + r^2\sqrt{(M^2 - a^2)}}{(r^2 + a^2)}$
Stability	Linear variation which turns to a constant value $C = \frac{c^2\left[\frac{4\pi}{c^4} - \propto \ln(\frac{1}{c^4 \pi r_+^4})\right]^{\frac{3}{2}}}{8.5\pi^2 r_+^3 (\frac{M}{2\pi r_+^2})^{\frac{3}{2}}}$	Linearly varies $C = 2\pi r\sqrt{(M^2 - a^2)}$

CHAPTER 10

CONCLUSION

A fascinating topic that touches on both the classical and quantum facets of gravity is black hole thermodynamics. Despite being fundamentally quantum in nature, a black hole's thermodynamic charges like entropy and temperature are connected to traditional characteristics like horizon size and surface gravity.

In this project , we analyze the thermodynamic characteristics of Kerr and Schwarzschild black holes. The horizon radius is used as the basis for the analysis of thermodynamic variables. Schwarzschild black hole, a solution to the Einstein's field equations, plays an important role in understanding the general theory of relativity. Kerr black hole is an uncharged black hole that rotates about a central axis. Slightly perturbed equilibrium solutions of the Schwarzschild black hole and Kerr black hole give rise to the black hole dynamics and they turned out to have a close analogy to the conventional thermodynamic laws. Also the selected systems are given logical corrections, and the variations of the thermodynamic parameters were simulated using MATLAB. The studies are further expanded to examine black holes' thermodynamic stability.

CHAPTER 11

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