**Project Report**

**On**

# ROMAN DOMINATION

***Submitted***

***in partial fulfilment of the requirements for the degree of***

**MASTER OF SCIENCE**

***in***

**MATHEMATICS**

*by*

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# CERTIFICATE

This is to certify that the dissertation entitled, **ROMAN DOMINATION** is a bonafide record of the work done by Ms. **BISMI ELIZABETH P.J** under my guidance as partial fulfillment of the award of the degree of **Master of Science in Mathematics** at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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# DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of **DHANALAKSHMI O M**, Assistant Professor, Department of Mathematics, St. Teresa's College(Autonomous), Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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# Chapter 1

# Basic Defnitions

## 1.1 Introduction

Graphs are used as mathematical models to analyze a variety of situations from the real world.Several mathematical disciplines, including group theory,matrix theory,probability,are closely related to graph theory. The 20th century has seen a fresh development in graph theory.Applications in the fields of biochemistry,electrical engineering (communication networks and coding theory),computer science(algorithms and computations) and operations research are among the key causes of this rise.

A rapidly growing topic of research in graph theory is domination. Chess was historically where domination – type problems first appeared. The eight queens puzzle,the queen domination problem,Queens and Knights and other chess related puzzles arouse the curiosity of players in the nineteenth centuary.

The defense strategies used to defend the Roman Empire under the rule of Emperor Constantine the Great served as the inspiration for the original study of Roman domination.He ordered that no more than two legions should be stationed in any city of the Roman Empire allowing for the possibility of sending one of the two legions to defend an attacked city.The mathematical idea of Roman Domination originated in this region of the Roman Empire.This project is a study on the double Roman domination and its relationship to both Roman domination and

**domination.Here we discuss certain properties of double Roman domination number,double Roman domination of some special graphs and ,Domination number of trees.**

## 1.2 Preliminaries

- **A Graph is an ordered triple G=(V(G),E(G),I(G),where V(G) is a vertex set,E(G) is a edge set disjoint from V(G) and I(G) is 'incidence' map which connects with each element E(G) an unordered pair of elements of V(G).Elements of vertex set are called vertices of G,and the elements of edge set are called edges of G.If for the edge e of G,$I_G$(e) = u,v,we have e =uv.**

- **If I(e)=u,v,then the vertices u and v are said to be the end vertices .A set of two or more edges of a graph G is said to be parellel edges if they have same pair of distinct ends.If e is an edge with end vertices u and v, write e=uv.An edge having the two ends are the same is called a loop at the common vertex.A vertex u is a neighbor of v in G,if uv is an edge of G.**

- **If both V(G) and E(G) are finite then the graph is a finite graph.A graph which is not finite is an infinite graph.We denote by n(G)and m(G) the number of vertices and edges of the graph G respectively.The number n(G) is called order of G and m(G) is the size of G.**

  **Let $G = (V, E)$ be a graph having order $n = |v|$ vertices.The open neighbourhood of a vertex $v \epsilon V$ is the set $N(V) = \{u : uv \epsilon E\}$and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Vertices $u \epsilon N(v)$ are called the neighborhood of v.**

- **A vertex with only one neighbor is a leaf and its neighbor is a support vertex.A support vertex having more than two leaf neighbors is a strong support vertex.**

- **The open neighborhood of a set $S \subseteq V$ is $N(S) = \cup v \epsilon SN(v)$,and the closed neighborhood of a set is the set $N[S] = N(S) \cup S$=$\cup v \epsilon SN[v]$.**

- The number of edges incident at v in G,is said to be the degree of the vertex v in G,and is denoted by deg(v).A loop at v is to be counted twice in computing the degree of.The minimum and maximum of the degrees of the vertices of a graph G is denoted by $\delta(G)$ and $\Delta(G)$ respectively.

- A vertex of degree 0 is an isolated vertex of G.A vertex of degree 1 is a pendant vertex of G, and the edge which is incident to such a vertex of G is a pendant edge of G.

  Example:

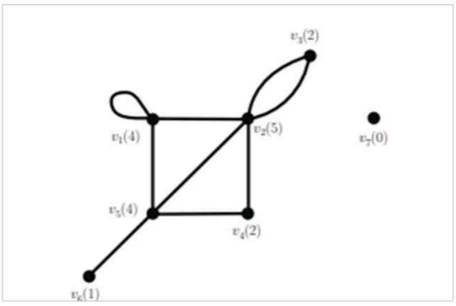


Figure 1.1: G1

In the graph G1,the numbers written in parentheses indicate the degree of the corresponding vertices.In G1 ,$v_7$ is an isolated vertex,$v_6$ is a pendant vertex and $v_3v_5$ is a pendant edge.

- A simple graph G is complete if every pair of vertices of G are joined by an edge in G.Any two complete graphs each on a set n vertices are isomorphic;such graph we denote it by $K_n$.

  Example

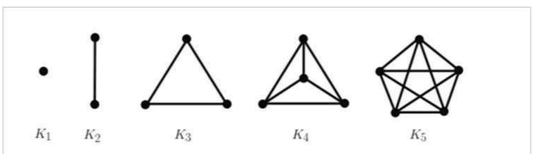


Figure 1.2: G2

- A simple graph with n vertices can have at most n(n-1)/2 edges.

**The complete graph $K_n$ has the maximum number of edges among all simple graphs with n vertices.On the other hand,a graph having no edge at all.Such a graph is a totally disconnected graph.**
**Example:**

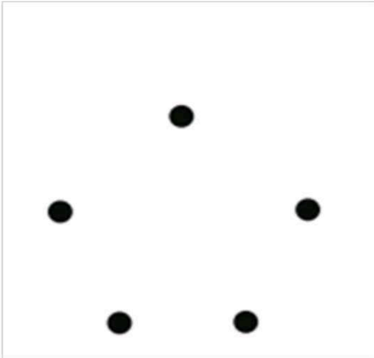

Figure 1.3: G3

- **If a vertex set is a singleton then the graph is trivial and it contains no edges.A graph is bipartite If the vertex set can be partitioned into two non empty subsets X and Y such that each edge of G has one end in X and the other in Y then the graph is a bipartite graph.The pair (X,Y) is called a bi partition of the bipartite graph.**

- **The bipartite graph G with bi partition $(X, Y)$ is denoted by $G(X, Y)$.A simple bipartite graph $G(X, Y)$ is complete if each vertex of X is adjacent to all the vertices of Y.If $G(X, Y)$ is complete with $|X| = p$ and $|Y| = q$, then $G(X, Y)$ is denoted by $K_{p,q}$.A complete bipartite graph of the form $K_{1,q}$ is called Star.**
**Example:**

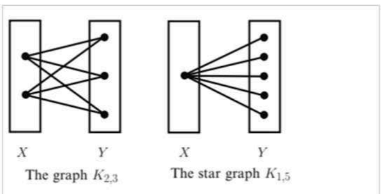


Figure 1.4: G4

- **A graph H is called a sub graph of G if V(H)** $\subseteq V(G)$,$E(H) \subseteq E(G)$ **and** $I_H$ **is the restriction of I(G) to E(H).If H is a sub graph of G,then G is said to be a super graph of H. A sub graph H of a graph G is a proper sub graph of G. If either** $V(H) \neq V(G)$ **or** $E(H) \neq E(G)$**. A sub graph H of G is an induced sub graph of G if each edge of G having its ends in V(H) is also an edge of H.**

- **A sub graph H of G is a spanning sub graph of G if** $V(H) = V(G)$**.The induced sub graph of G with vertex set** $S \subset V(G)$ **is called the sub graph of G induced by S and is denoted by G[S].Let E' be a subset of E and let S denote the subset of V consisting of all the end vertices in G of edges in E'.Then the graph(S,E',**$I_G(E')$ **is the sub graph of G induced by the edge set E' of G.It is denoted by G[E'].**

**Example:**

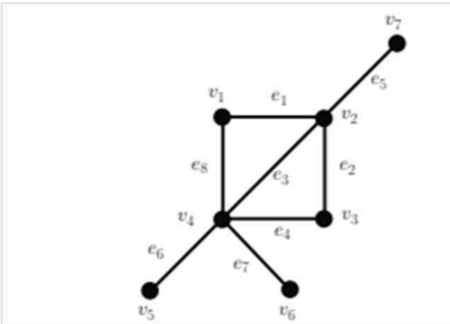


Figure 1.5: G5

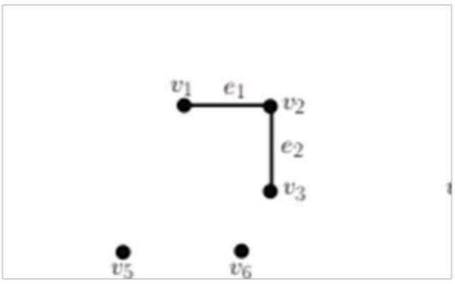


Figure 1.6: An induced sub graph of G5

## 1.3 Path ,cycle and tree

**In a graph G walk is an alternating sequence of vertices and edges W:$v_0e_1v_1e_2v_2....e_pv_p$ where $v_i - 1$ and $v_i$ are the ends of $e_i$;$v_0$ is the origin and $v_p$ is the terminus of W. If all the edges which appears in a walk are distinct,the walk is a trail. If all vertices are unique,then it is a path.A cycle is a closed trail in which the every vertices are distinct.A connected graph that have no cycles is a tree.We denote $P_n$ for path of order n,$C_n$ for cycle of length n.**

## 1.4 Connected graph

**There is a $uv$ path in if the two vertices u and v of a graph G is connected.The relation 'connected' is an equivalence relation on $V(G)$.Let $V_{i-1}, V_2 \ldots, V_W$ be the equivalence classes.The sub graphs $G[V_1], G[V_2], \ldots, G[V_W]$ are called the components of G.If $w = 1$,the graph G is disconnected.**

# Chapter 2

# DOMINATION,ROMAN DOMINATION AND DOUBLE ROMAN DOMINATION

## 2.1 DOMINATION

**Let S be a subset of a vertex set V.Then S is said to be a Dominating set if every vertex $u \epsilon V - S$ is adjacent to a vertex v in S.The minimum cardinality of a dominating set in G is called domination number $\gamma(G)$and dominating set of G with cardinality $\gamma(G)$ is called a $\gamma$set of G.If every vertex of $V - S$ has at least k neighbours in S then the subset S of V is called k dominating set.The k- domination number $\gamma_k$(G) is the minimum cardinality of a k- dominating set of G.**
**Example**

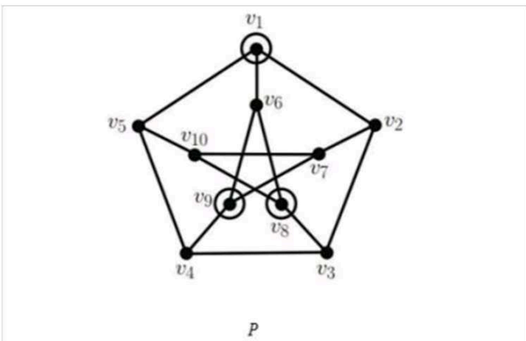


Figure 2.1: G10

**For the Peterson graph P, $\gamma(P)$ =3 and the dominating set is $S = \{v_1, v_8, v_9\}$.A subset $S \subset V$ is called a 2- dominating set if every vertex of $V - S$ has at least two neighbors in S.The minimum cardinality among all 2- dominating set of G is the 2-domination number of the graph G and is denoted by $\gamma_2(G)$.**

## 2.2 Types of Domination

### 2.2.1 Roman Domination

**Let$f : V \rightarrow${0,1,2}be a function having the property that for every vertex $v \epsilon V$ with $f(v) = 0$,there exist a neighbor u$\epsilon$N(v) with $f(v) = 2$.Such a function is called a Roman dominating function.The weight of a Roman dominating function is the sum $f(v) = \Sigma_{v\epsilon V} f(v)$.The Roman domination number of G is the minimum weight of a roman dominating function.It is denoted by $\gamma_{dR}(G)$.A $\gamma_R$-function of G is a Roman dominating function on G with weight $\gamma_R(G)$.**

**Given a graph G and a positive integer m,assume that $g : V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ is a function, and suppose that $\{v_1, v_2, \ldots, v_m\}$is the ordered partition of V induced by g,where $V_i$ =$\{v\epsilon V : g(v) = i\}$for i=$\{0, 1, 2, \ldots m\}$.So we can write $g(v_0, v_1, v_2, \ldots, v_m)$.**

**Example:**

**1) The Roman domination number of the peterson graph P is , $\gamma_R P_n = 6$ (assign the value 2 to the vertices $v_1, v_8, v_9$ and 0 to all the other vertices in previous example).**

**2) $\gamma_R(G) = \gamma(G) = n$ if and only if $G = K_n$**

**3) For the classes of path $P_n$ and cycles $C_n$, $\gamma_R(P_n) = \gamma_R(C_n) = \frac{2n}{3}$**

**4) $\gamma_R(K_{1,n-1}) = 2$ (assign the value 2 to the central vertex of the star $K_{1,n-1}$**

**5) If G is a graph of order n which contains a vertex of degree n-1,then $\gamma(G)$ =1 and $\gamma_R$(G) =2.The complete graph $K_n$ is an example**

**Roman Graph**

**A graph G is Roman if $\gamma_R$(G) =2$\gamma$.The complete graph $K_n$ is an example.**

**6)For cycles C$_3, C_4, C_5 \gamma_R(C_3) = 2, \gamma_R(C_4) = 3, \gamma_R(C_5) = 4$**

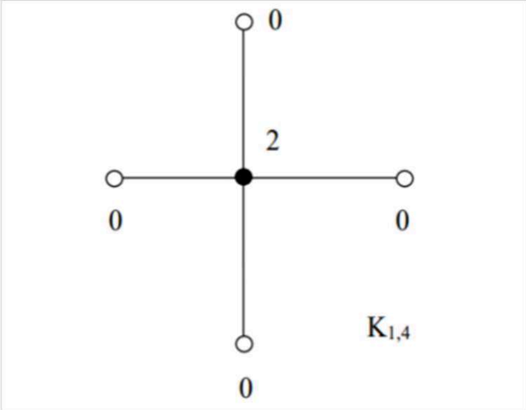


Figure 2.2: G11

### 2.2.2 Italian Domination

**An Italian domination function on a graph G is a function $f : V \rightarrow$ {0,1,2} which satisfies the property that for every vertex $v \epsilon V$ with $f(v) = 0$,the total weight of f assigned to the conterminous vertices of v is atleast 2. Easily,$\Sigma_{u \epsilon N(V)} f(u) \geq 2$. The weight of Italian dominating function f is defined as w(f) =f(v) =0, $\Sigma_{u \epsilon N(V)} f(u)$ The minimal weight among all of the Italian dominating functions on a graph G is called Italian domination number of G,denoted by $\gamma_I(G)$.**

## 2.3 Applications of Domination

**The subject graph theory has application in almost all branches of mathematics and everywhere there is an application of graph theory there will be an application of domination number also.Here are some of the areas where domination number can be applied.In security system,communication network,in biological network ,social network analysis.**

### 2.3.1 Security system

**If you need to place security system in a a particular area say in a mall every corridors and every corners of the mall is to be observed by a security guard and if we place security guards everywhere and in every corners that will increase the cost.So to have minimal cost effective**

security system,we have to place guards in such a way that the number of guards is minimized and every point is observed.Then we can find a dominating set.That is every corners will be a vertex and every other corners which upto which we can observe from this corner gives us an edge.That is the corridors become edges and we have to place security in such a way that every corridors and the corners that means a vertex at the extreme are observed.So we can find a dominating set.

### 2.3.2 Communication networks

In communication network also we use the concept of domination.If we need to broadcast an FM radio covering an entire district .If we a re placing a lot of broadcasting stations our cost effectiveness will be high.So we have to place minimum radio broadcasting stations and ensure that the broadcast reaches every house in that particular district .So it is clear that we are looking for a minimum dominating set.

## 2.4 Types of Roman Domination

- **Total Roman Domination**

  A total Roman domination function mapping is similar to Roman dominating function with a property that the vertices with non zero value into a subgraph which has no isolated vertex.The minimum weight of a total roman dominating function is called Total domination number.

  Example: $\gamma_{tR}(P_n) = \gamma_{tR}(C_n) = n$

- **Double Roman Domination**

  Robert A.Beeler,Teresa W.Haynes and Stephen T. Hedetniemi have defined double Roman Domination in 2016.They proposed a strong version of Roman domination

  A function $f : V \to \{0, 1, 2, 3\}$is a double Roman domination function of a graph G if the following conditions are satisfied.Let $v_i$ denote the set of vertices assigned I by a function f.

  If $f(v) = 0$, the vertex v must have at least two neighbors in $V_2$ or one neighbor in $V_3$.

**If$f(v) = 1$, then vertex v must have at least one neighbor in$V_2$ $\cup$V$_3$ The minimum weight of a double Roman dominating function is the double Roman domination number $\gamma_{dR}$(G),and a double Roman dominating function of G with weight $\gamma_{dR}$(G) is called $\gamma_{dR}-$ function of G.A double Roman dominating function f can be denoted as $f = (v_0, v_1, v_2, v_3)$ where $v_i = \{v \epsilon V(G) : f(v) = i\}$ for i=0,1,2,3.**
**Example:**
**The real benefit of double roman domination is for the star $K_{1,n-1}$,we can see that $\gamma_{dR}(K_{1,n-1}) = 3($ assign the value 3 to the central vertex of a star).Note that this is only one more than $\gamma_R(K_{1,n-1}) = 2$.**

**Consider the complete bipartite graph $K_{2,n-2}$ ,for $n \geq 4$,here the roman domination number is 3(in a partite set having size two,assign 2 to one of the vertices and 1 to the other,and 0 to all other vertices).But in order to achieve double Roman domination we have to increase the value of the vertex assigned 1 to 2.**

## 2.5 PROPERTIES OF DOUBLE ROMAN DOMINATION NUMBER

**Here showing that for any graph G,there exists a $\gamma_{dR}$ –function of G where no vertex assigned value 1,that is, $V_1 = \phi$ for some $\gamma_{dR}$ function of G.**

**PROPOSITION 1:**
**In a double Roman dominating function of weight $\gamma_{dR}$(G),no vertex need to be assigned value 1.**
**Proof:**
**Let f be a $\gamma_{dR}$ –function on a graph G.Suppose that for some v$\epsilon$V,f(v) =1.This means that there is a vertex u$\epsilon$N(v),such that either $f(u) = 2$ or $f(u) = 3$.If $f(u) = 3$ then we can achieve a double roman dominating function by reassigning a 0 to v,This results in a function with**

**strictly less weight than f , contradicting that f is a $\gamma_{dR}$ function.If $f(u) = 2$,then we can create a double roman domination function g defined as follows.$g(x) = f(x)$ for all x$\notin${u,v},$g(v) = 0$ and g(u) =3.This results in a double roman domination function with weight equal to f.**

**Double Roman Graph**

**A graph G is called double Roman if $\gamma_{dR}$(G) =3$\gamma$(G).Path $P_9$ is an example.**

**PROPOSITION 2:**

**Let G be a graph and f=($v_0$,$v_1$,$v_2$) a $\gamma_{dR}$ function of G.Then $\gamma_{dR}$(G)$\leq$ $2|V_1| + 3|V_2|$**

*Proof* :

**Let G be a graph and f=($v_0$,$v_1$,$v_2$) be a $\gamma_{dR}$function of G.**

**We define a function g=($v'_0$,$v'_1$,$v'_2$) as follows;$v'_0 = v_0$ ,$v'_2 = v_1$ and $v'_3$= $v_2$ .**

**Under G,every vertex with value 0 has an adjacent vertex with value 3,and no vertex have the value 1.**

**Therefor G is a double roman dominating function.**

**Thus $\gamma_{dR}$(G)$\leq 2|v'_2| + 3|v'_3| = 2|V_1| + 3|V_2|$.**

**Corollary 3:**

**For any graph G, $\gamma_{dR}$ (G) $\leq$ 2$\gamma_R$(G) with equality if and only if $G = \overline{K_n}$.**

**Proof:**

**Among all $\gamma_{dR}$function of G,let f=($v_0$,$v_1$,$v_2$) be one that minimizes the number of vertices in $V_1$.Since$\gamma_R$(G) =$|V_1| + 2|V_2|$,**

**By above proposition,we have**

$\gamma_{dR}$**(G)**$\leq 2|V_1| + 3|V_2|$

$\gamma_{dR}$**(G) =2**$\gamma_R$**(G) =**$\leq 2|V_1| + 4|V_2|$**,then**

**$\gamma_{dR}$(G)$\leq 2|V_1|+3|V_2|$ ,we must have $V_2$=$\phi$.V$_0$= $\phi$ must hold, and so V=$V_1$ since $V_1$ is minimized under f,we can see that no two vertices in G are adjacent,for otherwise,if u and v are adjacent,then the function f' which assigns a zero to u,a 2 to v,and a 1 to every other vertex is a $\gamma_{dR}$ function of G having a smaller number of vertices assigned 1 than f does.Thus $G = \overline{K_n}$.**

**Corollary 4:**

**If G is non trivial,connected graph and $f = (v_0, \mathrm{v}_1, v_2)$ is a $\gamma_{dR}$ function of G that maximizes the number of vertices in $v_2$, then $\gamma_{dR}$(G)$\leq$ $2\gamma_R$(G)$-|v_2|$.**

**Note:**

**By above corollary ,for a connected graph G,double roman domination gives double protection with strictly less tan double the cost of a roman dominating function.Next we see that the roman domination number is strictly smaller than the double roman domination number.**

**Proposition 5:**

**For every graph G,$\gamma_R(G)\leq\gamma_{dR}(G)$**

**Proof:**

**Let f=($v_0$,$v_1$,$v_2$) be any function of G,where $v_1 =\phi$**

**(by proposition 1 such function exists).If $\mathrm{V}_3 = \phi$, then every vertex in $V_3$ can be reassigned the value 2 and the resulting function will be a roman dominating function,that is $\gamma_R$(G)$<\gamma_{dR}$(G)**

**Assume that $V_3 =\phi$.Since $V_2\cup\mathrm{V}_3$ ,dominates G,it implies that $V_2\neq \phi$.Thus all the vertices are assigned either the value 0 or the value 2 and all vertices in $V_0$ must have atleast two nieghbors in $V_2$.In this case one vertex in $V_2$ can be reassigned the value 1 and the resulting function will be roman dominating function,**

**That is, $\gamma_R$(G)$<\gamma_{dR}$(G)**

**Corollary 6:**

**If f=($v_0$,$v_1$,$v_2$) is any $\gamma_{dR}$ function of a graph G,then**

$\gamma_R$**(G)**$\leq 2(|V_2| + |V_3|) =\gamma_{dR}$**(G)**$-|v_3|$

**Corollary 7:**

**For any non trivial connected graph G, $\gamma_R$(G)$<\gamma_{dR}$(G)$< 2\gamma_R$(G).**

# Chapter 3

# Double Roman Domination and Domination

**We first give some lower and upper bounds on the double roman domination number in terms of the domination number.**

**Proposition 8:**

**For any graph G,2$\gamma$(G) $\leq\gamma_{dR}$(G)$\leq 3\gamma$(G).**

**Proof:**

**For the lower bound,let $f =$ (v$_0$,$v_1$,$v_3$)$bea\gamma_R$ function of a graph G.Let S be a $\gamma$(G) set.Not that $(\phi, \phi, S)$ is a double roman dominating function.This yields the upper bound of $\gamma_{dR}$(G)$\leq 3\gamma$(G).Further $V_2\cup$V$_3$ is a dominating set for G.Thus $\gamma$(G)$\leq |V_2| + |V_3|$.Using this observation we can obtain the lower bound, $\gamma_{dR}(G) = 2|V_2| + 3|V_3| \geq 2(|V_2| + |V_3|) \geq 2\gamma(G)$.**

**Definition:**

**The 2 domination number $\gamma_2$(G) equals the minimum cardinality of a set such that every vertex in V-S is adjacent to at least two vertices of S.**

**Proposition 9:**

**For any graph G,$2\gamma(G) = \gamma_{dR}(G)$ if and only if $\gamma(G) = \gamma_2(G)$**

## 3.1 ROMAN DOMINATION NUMBER AND ITALIAN DOMINATION NUMBER

**We first state our upper and lower bounds of the Italian domination number in terms of Double roman domination number.**

**Theorem 10:**

**For every graph G,** $\gamma_{dR}(G)/2 \leq \gamma_I(G) \leq 2\gamma_{dR}(G)/3$

**Proof:**

**Let** $f = (\mathbf{v_0}^f, \mathbf{v_1}^f, \mathbf{v_2}^f)$ **be an** $\gamma_I$**(G)arbitrary function.**

**Then the function** $g = (\mathbf{v_0}^f, \phi, \mathbf{v_1}^f, \mathbf{v_2}^f)$ **is a double roman domination function for G.**

**Thus** $\gamma_{dR}$**(G)**$\leq 3|\mathbf{V}_2^f$— **+ 2**—$V_1^f$— $\leq 4|\mathbf{V}_2^f$— **+ 2**—$V_1^f$— **=2** $\gamma_I$**(G)as desired.We next establish the upper bound of the theorem.**

**Let** $g = (v_0^g, \phi, v_2^g, v_3^g)$ **be an arbitrary** $\gamma_{dR}$**(G)function.**

**Then** $h = (v_0^g, \phi, v_2^g, v_3^g)$ **is an Italian dominating function for G.**

**Thus** $\gamma_{dR}$**(G)**$\leq w(h) = 2|V_3^g| + |V_2^g| =\gamma_{dR}$**(G)**$-(|V_3^g| + |V_2^g|)$**.**

**On the other hand, since** $\mathbf{V}_2^g \cup V_3^g$**is a dominating set for G ,** $\gamma$**(G)**$\leq |V_3^g| + |V_2^g|$**.**

**Therefore** $\gamma_I$**(G)**$\leq\gamma_{dR}$**(G)**$-\gamma$**(G)**

**By proposition 8,we have**

$\gamma_{dR}(G) \leq 3\gamma(G)$

**and so** $\gamma_I(G) \leq \gamma_{dR}(G) - \gamma(G) \leq \gamma_{dR}(G) - \gamma_{dR}(G)/3 = 2\gamma_{dR}(G)/3$

**Note:If** $\gamma_{dR}(G)/2 = \gamma_I(G)$ **and** $4f = (v_0^g, \phi, v_2^g, v_3^g)$ **be an arbitrary** $\gamma_I$**(G) function then the following proof of proposition 8, we get** $V_2^f = \phi$**.Thus** $V_1^f$ **is a 2 dominating set for G and so** $\gamma_2(G) \leq |V_1^f| = \gamma_I(G)$**.Since always** $\gamma_I(G) \leq \gamma_2$**(G) we obtain that** $\gamma_I(G) = \gamma_2(G)$

**Corollary 11:**

**If** $\gamma_{dR}(G)/2 = \gamma_I(G)$ **then** $\gamma_I(G) = \gamma_2(G)$**Furthermore** $V_2^f = \phi$ **for every** $\gamma_I$**(G) function** $f = (v_0^f, v_1^f, v_2^f)$**.**

**Corollary 12:**

**For a graph G,** $\gamma_I(G) = 2\gamma_{dR}(G)/3$ **if and only if** $\gamma_{dR}(G) = 3\gamma(G)$ $\gamma_I(G) = 2\gamma(G)$**.**

**Corollary 13:**

**If for a graph G,** $\gamma_I(G) = 2\gamma_{dR}(G)/3$ **,then for any** $\gamma_{dR}(G)$ **function** $f = (v_0^f, {}_1^f, v_2^f) = \phi$**.**

## 3.2 DOUBLE ROMAN DOMINATION OF SPECIAL GRAPHS

**Proposition 14:**

**For n** $\geq 1 \gamma_{dR}(P_n) = \begin{cases} n, & \textbf{ifn} \equiv 0 \pmod 3 \\ n+1, & \textbf{ifn} \equiv 1 or 2 \pmod 3 \end{cases}$

**Proof:**

**Let P:**$\mathbf{v}_1, v_2 \dots v_n$ **be a path of order n. Define** $h : V(P_n) \to \{0, 1, 2, 3\}$ **by** $h(v_{3i-1}) = 3$ **for 1** $\leq i \leq n/3$**and**$h(x) = 0$**otherwise ifn**$\equiv 0 \pmod 3$ **by** $h(v_n) = 2$**,** $h(v_{3i-1}) = 3$ **for 1** $\leq i \leq (n-1)/3$ **and** $h(x) = 0$ **otherwise. When** $n \equiv 1 \pmod 3$ **by** $h(v_n) = 3$**,** $h(v_{3i-1}) = 3$ **for 1** $\leq i \leq (n-2)/3$ **and h(x) =0**$otherwise if$**n**$\equiv 2 \pmod 3$

**Here h is double roman dominating function of** $P_n$ **of weight n if** $n \equiv 0 \pmod 3$**.**

**Thus**

$$\gamma_{dR}(\mathbf{P}_n) = \begin{cases} n, & \textbf{if n} \equiv 0 \pmod 3 \\ n+1, & \textbf{if n} \equiv 1 or 2 \pmod 3 \end{cases}$$

**Now we prove the inverse inequality .The proof is by induction on n. The statement holds for all paths of order n** $\leq 4$**.For the inductive hypothesis, let n** $\geq 5$ **and suppose that for every path of order less than n the result is true.**

**Assume f=(**$\mathbf{v}_0, v_1, v_2, v_3)$ **is a** $\gamma_{dR}$ **function of** $\mathbf{P}_n so that V_1 = \phi$**.**

**First let f(**$\mathbf{V}_n) = 0$ **Then we have f(**$\mathbf{V}_{n-1}) = 3$**.**

**Define g:V (**$\mathbf{P}_{n-3}) \to \{0, 1, 2, 3\}$ **by g (**$\mathbf{v}_{n-3}) = min\{3, f(V_{n-2}) + f(V_{n-3})$**and g (**$\mathbf{v}_i) = f(v_i)$**for i =1,. . .,n-4.**

**Clearly g is a double roman dominating function of** $\mathbf{P}_{n-3} of weight(f)-3$**.It follows from the inductive hypothesis that**

$$\gamma_{dR}(\mathbf{P}_n) = (f) = (g) + 3 \geq \gamma_{dR}(\mathbf{P}_{n-3}) + 3 \geq \begin{cases} n, & \textbf{if n} \equiv 0 \pmod 3 \\ n+1, & \textbf{if n} \equiv 1 or 2 \pmod 3 \end{cases}$$

**If f(**$\mathbf{V}_n) + f(V_{n-1}) \geq 3$**, then the function g :** $\mathbf{V}(\mathbf{P}_n) \to \{0,1,2,3\}$ **defined by** $g(v_n) = 0$**,** $g(v_{n-1}) = 3$**and** $g(x) = f(x)$ **otherwise, is clearly a** $\gamma_{dR}$ **function of** $P_n$**.As above we obtain the inverse inequality. Let** $f(v_n)+f(v_{n-1}) \leq 2$**.Then we must have** $f(v_n) = 2$ **and** $f(v_{n-1}) = 0$ **and this implies that f(**$\mathbf{v}_{n-1}) \geq 2$**. If** $f(v_n) + f(v_{n-1}) \geq 3$**, then the function g:** $\mathbf{V}(\mathbf{P}_n) \to \{0,1,2,3\}$**defined by g(**$\mathbf{v}_n) = g(v_{n-2}) = 0, g(v_{n-1}) = 3, g(v_{n-3}) = max\{2, f(v_{n-3})\}$ **and g(x) = f(x) otherwise, is clearly a** $\gamma_{dR}$**-function of** $\mathbf{P}_n$ **and we can obtain the inverse inequality as above. Let f(**$\mathbf{v}_{n-2}) + f(v_{n-3}) \leq 2$**.Then we have f(**$\mathbf{v}_{n-2}) = 2$ **and f(**$\mathbf{v}_{n-3}) = 0$**.To double Roman dominate** $\mathbf{v}_{n-3}$**,we must have f(**$\mathbf{v}_{n-4}) \geq 2$**.Define g:**$\mathbf{V}(\mathbf{P}_n) \to \{0,1,2,3\}$**by g(**$\mathbf{v}_{n-4}) = g(v_{n-3}) = 3$, **g(**$\mathbf{v}_n) = g(v_{n-2}) = g(v_{n-3}) = 0$ **and g(x) = f(x)otherwise. It is easy to see that g is a** $\gamma_{dR}$**-function of** $\mathbf{P}_n$ **and as above we can obtain the inverse inequality. Thus**

$$\gamma_{dR}(\mathbf{P}_n) = \begin{cases} n, & \textbf{if n} \equiv 0 \pmod 3 \\ n+1, & \textbf{if n} \equiv 1 or 2 \pmod 3 \end{cases}$$

**Proposition 15:**

$$\gamma_{dR}(\mathbf{C}_n) = \begin{cases} n, & \textbf{if n} \equiv 0,2,3,4 \pmod 6 \\ n+1, & \textbf{if n} \equiv 1,5 \pmod 6 \end{cases}$$

**Proposition 16:**

**Let G be a connected graph of order n**$\geq 3$**.**

**Then** $\gamma_{dR}(G) = 3 if and only if \Delta(G) = n-1$.

# Chapter 4

# UPPER BOUNDS IN TERMS OF ORDER AND DEGREE

**The upper bound is sharp for particular trees,specifically stars,while the lower bound is sharp for the family of complete bipartite graphs $K_{2,k} for k \geq 2$.Next finding demonstrates that lower bound for trees can be enhanced.The tree with exactly two adjacent non – leaf vertices, one of which is next to r leaves and the other to s leaves is known as double star $S_{r,s}$.Let $T_v$ be the subtree of T rooted at v and its descendants in T for a tree with roots at r and v of type T.Let T-$T_v$ stands for the tree rooted at r generated by deleting the subtree $T_v$ from T and having its roots at r.The edges and vertices of $T_v$ and the edge from v to its parent in T are deleted from T to form T-$T_v$ which is a tree.**

**Theorem 17:**

**If T is a non – trivial tree, then $\gamma_{dR}$(T)$\geq 2\gamma_{dR}$(T)+ 1.**

**Corollary 18:**

**For any tree T,2 $\gamma(T) + 1 \leq \gamma_{dR}$(T) $\leq 3\gamma(T)$**

**Note:**

**The corona G $\circ K_1$ is the graph formed from G by adding a new vertex**

**v' and edge v'v for each vertex v** $\epsilon V(G)$.

**Proposition 19:**

**An ordered pair (a,b) is realizable as the domination number and the double roman domination number of some trivial tree if and only if 2a+1**$\leq b \leq 3a.$

**Proof:**

**Let T be a tree with** $\gamma(T)$ **=a and** $\gamma_{dR}(T) = b$.**Then by corollary 18 2a+1**$\leq b \leq 3a. For b = 2a+1$ **consider corona of star** $K_{1,t}$ **for t**$\geq 1$, **To check** $\gamma(K_{1,t})\circ$ $\mathbf{K}_1 = t+1$ **and** $\gamma_{dR}(K_{1,t})\circ$ $\mathbf{K}_1 = 2t+3 = 2\gamma(K_{1,t}\circ\mathbf{K}_1)+1$.

**Assume b**$\geq 2a+2$.

**Let T be a tree generated from a subdivided star** $K_{1,a}*$ **by selecting b–(2a+2) support vertices of** $K_{1,a}*$ **and adding another leaf neighbor to each of them.**

**Thus T has b-2a-2 strong support vertices.Again to check that** $\gamma(T)$**=a(set of support vertices generate a** $\gamma$**-set of T).**$\gamma_{dR}(T)$ **=b,each b-2a-2 strong support verices must be assigned a 3 under any double roman dominating function of T.Assigning a zero to support vertices with one leaf , 2 to adjacent leaf and the centre.It is simple to check that these functions are in fact minimum.**

**Hence we have** $\gamma_{dR}(T) = 3(b-2a-2)+2(a-(b-(2a+2))+2$

$= 3b-6a-6+2a-2b+4a+4+2 = b.$

**Note:**

**If G is a graph of order n,and has no isolated vertices,then** $\gamma \leq n/2$.

**Since we have shown that for any graph G,** $_{dR}(G) \leq \gamma(G)$ **then for all graphs of order n,having no isolated vertices,** $\gamma_{dR}(G) \leq 3n/2$

**Theorem 20:**

**If T is a tree of order n**$\geq 3, then \gamma_{dR}(G) \leq 5n/4$

**Corollary 21:**

**If G is a connected graph of order n**$\geq 3, then \gamma - d_R(G) \leq 5n/4.$

**The bound of Theorem 20 shows that the trees attaining the bound.For this,define the family F of trees as:**
**Let T be a tree and v be its non –leaf vertex .Define f(T,v) be a tree obtained from T by adding a $P_4$ with a support vertex u to T through the edge uv.Let the smallest family of graphs be F:**
**F contains $P_4$ and if T$\epsilon$ F and a non –leaf vertex of T,then f(T,v)$\epsilon F$ or we can say a tree T in F can be built from k copies of $P_4$ by adding k-1 edges incident to support vertices of the $k_{P,4}$ to connect the graph.**

# CONCLUSION

**Domination is currently regarded as one of the main ideas in graph theory,and the fact that it has so many different applications in areas biological networks,distributed computing,social networks and web graphs helps to explain why interest has grown.**

**In graph theory there are two variations on the idea of dominance :Roman domination and double roman domination.Double roman domination doubles the protection by ensuring that any attack can be defended by atleast two legions.Under roman rule only two legions can be stationed at once.But as we seen above the ability to send three legions to a specific site offers a degree of defense that is more powerful and adaptable for less money than was one thought to be necessary.**

**Problems with selecting sets for representatives ,monitoring communication or electrical networks also involve concepts of domination.Another interesting field of application is social networking;if we consider the vertices as people and two vertices are connected if there exists a friendship between the two people,then a dominating set is the set of people who knows everyone.**

# LITERATURE REVIEW

**In his work"Roman domination" by Linfeng XU , published in 1999,Ian Stewart discussed a strategy of emperor constantine's defense of the Roman empire approach.Cockayne et al.(2004) introduced the idea of Roman domination in graphs as a result of this article.The essential structure of Roman domination in graphs was introduced in this study.The existing study focuses on special properties of this concept and this paper helps to build a foundation for understanding various advanced problems.**

**In 2020 Mustapha Chellali, Nader Jafari Rad, Seyed Mahmoud Sheikholeslami  Lutz Volkmann had studied the results on the Roman domination number as well as those when the structure of the graph is modified by the addition of edges/vertices or removing edges/vertices in the paper Roman Domination in Graphs.**

**In 2016, Robert A. Beeler a, Teresa W. Haynes a b, Stephen T. Hedetniemi studied about double Roman domination and show its relationship to both domination and Roman domination.**

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