PROJECT REPORT

ON

THERMODYNAMIC STUDY OF BARDEEN BLACKHOLE AND TAUB NUT CHARGED BLACKHOLE

SUBMITTED BY

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REGISTER NO. : AM20PHY006

in partial fulfillment of

the requirements for award of the postgraduate degree in physics

DEPARTMENT OF PHYSICS AND CENTRE FOR RESEARCH

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M.Sc. PHYSICS PROJECT REPORT

Name :

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Year of Work :

This is to certify that the project "**THERMODYNAMIC STUDY OF BARDEEN BLACKHOLE AND TAUB NUT CHARGED BLACKHOLE**" is the work done by **Henna Mary Jose**

Dr. Priya Parvathi Ameena Jose — Head of the Department

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Submitted for the University Examination held in St Teresa's College,Ernakulam

Date : ……………………………………………………

Examiners 1) …………………………………………………

2) ………………………..

DEPARTMENT OF PHYSICS

ST TERESAS COLLEGE (AUTONOMOUS), ERNAKULAM

CERTIFICATE

This is to certify that the project report entitled "**THERMODYNAMIC STUDY OF BARDEEN BLACKHOLE AND TAUB NUT CHARGED BLACKHOLE"** is the bonafide work done by **Ms. Henna Mary Jose** (Reg. No: AM20PHY006) under the guidance of **Dr.THARANATH .R** ,Assistant Professor, Department of Physics, Aquinas College, Edacochin and **Dr.SUSAN MATHEW** ,Assistant Professor, Department of Physics and Centre for Research, St. Teresa's College, Ernakulam in partial fulfilment of the the award of the Degree of Master of Science in Physics, St. Teresa's College, Ernakulam affiliated to Mahatma Gandhi University, Kottayam

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DECLARATION

I, hereby declare that the project work entitled "**THERMODYNAMIC STUDY OF BARDEEN BLACK HOLE AND TAUB NUT CHARGED BLACK HOLE**" is a record of an original work done by me under the guidance of **Dr.THARANATH R**, Assistant Professor, Department of Physics ,Aquinas College Edacochin, and **Dr .SUSAN MATHEW**, Assistant Professor, Department of Physics and centre for Research, St Teresa's College, Ernakulam in the partial fulfilment of the requirements for the award of the Degree of Master of Physics. I further declare that the data included in the project is collected from various sources and are true to the best of my knowledge.

Henna Mary Jose

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ABSTRACT

Black holes are the robust predictions of Einstein's General Relativity. The study of the thermodynamics properties of black hole is a major research theme of contemporary theoretical physics. In this project work we have studied the thermodynamics of Reissner–Nordström , Bardeen black hole and Taub NUT blackhole. We have derived the thermodynamic quantities such as Mass, Temperature, Entropy, Heat Capacity and Free Energy and plotted their variations with respect to entropy. Bardeen black hole shows a smooth variation of temperature with respect to entropy, which excludes the chance of first order phase transition. Furthermore, the discontinuity in heat capacity for a particular value of entropy in Bardeen black hole shows the presence of a second order phase transition. For Taub-NUT, heat capacity is negative and it indicates that the black hole is unstable and the free energy falls to negative above a certain value.

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CHAPTER-1

INTRODUCTION

For more than two hundred years, Newton's theory of gravitation was accepted as the valid theory to describe the gravitational force. The framework, provided by Newton was considered as extremely successful to describe the motion of celestial objects. However, there were several incongruities in this theory such as: it could not explain the perihelion precession of Mercury; also, it could not explain how the gravitational force comes into play between two objects which are very far from each other and not connected via any medium.

In spite of these limitations, the theory, proposed by Newton, was widely regarded as the appropriate theory for gravity as it was highly successful in describing the motion of the objects under the influence of gravity. Thus, in 1915, when Einstein came up with General Relativity (GR), describing gravity in terms of the geometry of the spacetime itself, it took a while for people to accept it as the better theory for describing gravity. However, the famous experimental test during the solar eclipse of 1919, carried out by Arthur Eddington and Frank Dyson, made the theory famous overnight. A few years ago, in 2015, we have completed hundred years since Einstein proposed the theory of general relativity (GR), which is now considered as one of the most wonderful theories of physics ever proposed in the human history. Not only the theory can explain several observational phenomena, like the bending of light, perihelion precession of Mercury, gravitational lensing, gravitational redshift etc., but also the theory is sublime in its mathematical foundation. The recent discovery of gravitational waves in 2016, which was also the prediction of Einstein's GR, has added another feather to the crown of this theory. Time and again, this theory has proved to be the most viable theory for gravity.

A few months after Einstein's new formulation of GR, in 1916, Karl Schwarzschild found the solution of Einstein's equation for a point mass. About the same time, Johannes Droste independently arrived to the same solution. The solution had a strange behaviour which, nowadays, known as the Schwarzschild radius where the Einstein's equation becomes infinite. In 1958, David Finkelstein identified the surface area of the Schwarzschild radius as the event horizon as it acts as a one way membrane and any causal curve can cross it only in one direction. This result paved the idea of black hole (BH) and the research on black holes became one of the most active areas in theoretical physics till the date.

Later in 1963, Roy Kerr discovered the solution of the rotating black holes– known as the Kerr black hole. Meanwhile, the no-hair theorem emerged, which states that the stationary black holes can be completely described by only three parameters: mass, charge and the angular momentum. For a long time, some people doubted the existence of black holes. However, the recent discovery of gravitational waves abolished the doubt regarding the real existence of black hole. A year ago, in 2019, the first ever image of black hole and its surroundings was published, which was observed earlier by Event Horizon Telescope in 2017. Now, in the very last month of May, 2022 astronomers have unveiled the first image of the supermassive black hole at the centre of our own Milky Way galaxy observed by Event Horizon Telescope.

In the decade of 1970's, several remarkable works came up which added new perception to the study of black hole theory. These new results have shown the connection of gravity with the thermodynamics. In one of his famous works, Hawking had shown some important results for black holes in general relativity. Most importantly, the paper provided the area increase theorem of black hole horizon, which states that the horizon area of a black hole horizon always increases when the specific energy condition is satisfied. Bekenstein realized that the black holes must have entropy and he ascribed the entropy of the black hole to be proportional to its horizon surface area. Thereafter, the four laws of blackhole mechanics were shown by Bardeen, Carter and Hawking, which had an astonishing similarity with the four laws of thermodynamics. However, the authors in this paper refrain themselves from claiming it as the thermodynamic laws of black holes. Instead, they claimed it as an analogy with the conventional thermodynamics.

In fact, this analogy became a robust correspondence with the thermodynamics when Hawking revealed that the black holes can radiate when quantum effects are taken into the consideration. This radiation was later famously known as the Hawking radiation. Although Hawking initially tried to disapprove Bekenstein's idea about black hole entropy, this work by Hawking justifies the earlier claim by Bekenstein and also fix the proportionality constant of black hole entropy with the horizon area as 1/4 in natural unit.

In the meantime, Fulling Davies and Unruh had shown that the accelerating observer observes thermal radiation in the Minkowski vacuum whereas the inertial observer does not. The temperature of the Unruh particles have the same form as of the Hawking temperature, except the surface gravity of black hole horizon is replaced by the acceleration of the observer. This radiation was later known as the Unruh radiation. The Unruh radiation and the Hawking radiation are equivalent on the basis of Einstein's equivalence principle. These were the stepping stone which laid the foundation of black hole thermodynamics. Since then, there have been numerous works in the direction of black hole thermodynamics and it became one of the most high-yielding domains for the theoretical physicists over the years.

Later several thermodynamic features were found in black hole thermodynamics and the earlier analogies (the area of the black hole horizon as the entropy, surface gravity of the horizon as the temperature etc.) were firmly identified as the physical thermodynamic parameters of black holes.

Phase transition is another important aspect of thermodynamics, which is also found in black hole thermodynamics as well and it has been studied for several decades. There are several types of phase transitions which are present in black hole thermodynamics. It was first introduced by Davies who argued that black holes undergo a second order phase transition when it passes through a point, which known as the Davies' point, where the heat capacity diverges. However, later Kaburaki et. approved that the Davies's point is not the critical point. Instead, it is a turning point, where the stability changes.

Another type of black hole phase transition was found in the work of Hawking. It was found that a black hole in AdS space makes transition to a no-black-hole state (or radiation) at a critical temperature. In addition, the transition of black holes from a non-extremal to an extremal one is also been found out as a phase transition of black holes, which is known as the extremal phase transition.

1.1 GENERAL THEORY OF RELATIVITY

General theory of relativity (GTR), published by Albert Einstein in 1915, is the current description of gravitation in modern physics. The general theory of relativity is proposed after special theory of relativity (STR). GTR comes to the main frame that STR must be modified to include the presence of gravity. This is because the STR is similar to dealing Newton's equations of motion without considering friction. When it comes to the realistic situation gravity is inevitable so a generalized concept of relativity is needed.

The Einstein way of describing gravity is to avoid the notion that is a force and it is to find suitable non-Euclidean space time geometry and matter under no force moving in straight line trajectories with uniform speed as measured in terms of the rules of the new geometry. General relativity pictures gravity as a warping of space time due to the presence of a body of matter. An object nearby experiences an attractive force as a result of this distortion, much as a marble roll towards the bottom of depression in a rubber sheet. According to J. A. Wheeler, spacetime tells mass how to move, and mass tells space time how to curve.

The principle of equivalence is central to general relativity. An observer in a closed laboratory cannot distinguish between the effect produced by a gravitational field and those produced by an acceleration of the laboratory. To understand this, consider two observers. Let of them be in a room which is at rest in a uniform gravitational field at the surface of earth and the second observer be in a room inside a rocket accelerating at g. Now if the former drops a pebble, it will accelerate towards the floor with acceleration g. This would be the same situation happening to the second observer. i.e. in both cases the observers are unable to distinguish the two situations by local experiments. This cancellation of gravitational fields by inertial forces is applicable to all freely falling systems. So that no local experiments can be distinguish between uniform inertial and gravitational accelerations.

1.2 THE SPACE TIME METRIC

We use a coordinate system to map the space around us. Consider the collection of all possible events that occur in the universe- that can be happened in the past, events happening now or the one that will happen in the future. The collection of these events is called spacetime. Let us assume that we use a linear coordinate system, so that we can use linear algebra to describe it. Physical objects can then be described in terms of the basis-vectors belonging to the coordinate system.

Let (x, y, z) denote a Cartesian coordinate system and that would give the spatial location of the event and t, the time measured by an observer O at rest in an inertial frame , that is an observer who is acted on by no force. Let two neighbouring events in space and time be labelled by the coordinates (x, y, z, t) and (x + dx, y + dy, z + dz, t + dt). The square of the ‘distance’ between the two events is given by

$$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2$$

The distance ds is invariant under Lorentz transformation, in the sense that if another inertial observer O’ using a different coordinate system (x’, y’, z’, t’) to measure this distance will find the same answer. The above equation is the one commonly used in special theory, when we transform from special theory to general theory, the presence of gravitation requires a more complicated form. This can be conveniently written as,

$$ds^2 = \sum_{i,k=0}^{3} g_{ik} dx^i dx^k$$

with i =1, 2, 3 representing the three space coordinates and i = 0 the time coordinate. Here g_{ik} are the components of a second rank tensor. The matrix elements representing the tensors are the coefficient functions that multiply the differentials in the metric. The expressions for ds^2 is referred to as the metric. Now for convenience we can drop the summation symbol Σ by using Einstein’s summation convention (For example, $\sum_{i=0}^{3} A_i B^i$ can be written as $A_i B^i$) the rule being that, whenever an index appears once as a subscript and once as a superscript in the same expression, it is automatically summed over all the values (here it is from 0 to 3) . Thus, we can write

$$ds^2 = g_{ik} dx^i\, dx^k .$$

1.3 EINSTEIN’S FIELD EQUATIONS

The ground breaking discovery of General theory of relativity was that it describes gravity as the effect of curvature in the fabric of spacetime geometry, a 4- dimensional picture that unifies space and time into a single framework. Coming to the field equations,

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi G}{C^4}\,T_{ab},$$

where Rab is the Ricci curvature tensor, $Tab$is the stress-energy tensor, gab is the metric tensor R is the Scalar curvature, G is the Newtonian constant of gravitation and c is the speed of light in vacuum. It consists of a coupled system of ten partial differential equations. The right hand side of the equation describes the distribution of the mass and energy, whereas the left-hand side describes the geometry of space time.

In other words, Einstein's field equations give how mass and energy curve spacetime give rise to gravity. The right-hand side of the equation would be zero if there are no matter fields. Those are known as vacuum Einstein field equation. Einstein, using his field equations, predicted the perihelion motion of planet mercury; explained the bending of light in the vicinity of sun and the gravitational red shift.

1.4 BASIC THERMODYNAMICS

Thermodynamics is a branch of physics that deals with heat, work, and temperature, and their relation to energy, radiation, and physical properties of matter. It describes how thermal energy is converted to and from other forms of energy and how it affects matter. Thermal energy is the energy a substance or system has due to its temperature .The study of thermodynamics deals with systems having large number of particles enclosed in a surrounding. It can be a liquid in vacuum flask or a canister of gas and the walls of the system prevents the exchange of energy. An ideal insulating wall is purely a theoretical notion. So when coming to the real experiments, the wall would not be perfect insulators. The observations of these experiments are now formulated to be the laws of thermodynamics. The study of thermodynamics can follow two paths, classical thermodynamics and statistical thermodynamics. The former approach concentrates on the gross behaviour of matter, evaluate measurable properties

1.4.1 Thermodynamic equilibrium

When the energy flow stops between the systems and all measurable properties are independent of time, the combined system is said to be in equilibrium. Consider an example to describe the state equilibrium. Consider two closed canisters, each having a piston attached to it. Let the piston be pinned so as to make the system rigid. Let the pistons be attached by a rod in common so that moving one piston moves the second. If we let the pin that place the piston in place to move, the two pistons move. Both the pistons may not have the same pressure and does not move alike, but after sometime the pistons would stop moving, making the pressure on the two pistons equal. This can be described as a state of equilibrium. There are three different aspects of equilibrium between systems. When there is no unbalanced force in the interior of a system and also none between a system and surroundings the system is said to be in a state of mechanical equilibrium.

When a system in mechanical equilibrium does not tend to undergoes a spontaneous change of internal structure such as chemical reaction or a transfer of matter from one part of the system to another such as diffusion or solution, however slow then it is said to be in a state of chemical equilibrium. Thermal equilibrium exists when there is no spontaneous change in the coordinates of a system in mechanical and chemical equilibrium when it is separated from its surroundings by a diathermic wall. All parts of system are at the same temperature and this temperature is the same as that of surroundings

When these conditions are not satisfied a change of state will take place until thermal equilibrium is reached. When the conditions for all three types of equilibrium are satisfied the system is said to be in a state of thermodynamic equilibrium.

1.4.2 Laws of thermodynamics

Fowler discovered the phenomenon of thermal equilibrium in 1935. All the three laws of thermodynamics were discovered before 1935 thus newly discovered law was named the Zeroth law of thermodynamics.

Zeroth law states that if systems A and B are separately in thermal equilibrium with C, then systems A and B are in thermal equilibrium with each other. The law indicates that when system A and B are placed in contact with each other, no property of either system A or B changes with time which indicates that no energy flows between them if the system are in same temperature.

First Law of Thermodynamics states that a small amount o heat given to a system is partly used in doing external work and partly to increase the internal energy. Consider a quantity called the internal energy of the system, denoted by the symbol U. We choose U to ensure that the total energy of the system is constant. To preserve the law of conservation of energy we write the change in internal energy from its initial value U_i to its final value U_f as

$$\Delta U = U_f - U_i$$

$$\Delta U = W + Q$$

where Q is defined as the heat which is added to the system, W is the work done on the system. The change in the internal energy is the sum of the work done on the system and the heat added. The first law of thermodynamics says that energy is conserved if heat is taken into account.

The limitation of the first law is that it doesn't tell us about the direction in which the process occurs. This limitation was solved by the discovery of the second law. Second Law of thermodynamics states that Energy prefers to flow from a body with higher temperature to one with a lower temperature. Heat will not flow spontaneously from a cold object to a hot object.

Statement of the second law due to Clausius: It is impossible to construct a machine which operating in a cycle produces no other effect than to transfer heat from a colder to a hotter body. It is not possible for heat to flow from a colder body to a warmer body without any work having been done to accomplish this flow. Another statement of the second law is due to Kevin and Planck: It is impossible for an engine, operating in a cycle to take heat from a single reservoir, produce an equal amount of work and to have no other effect. It is impossible to extract an amount of heat Q_H from a hot reservoir and use it all to do work W. Some amount of heat Q_C must be exhausted to a cold reservoir. This precludes a perfect heat engine.

Now, before going through the statement of third law, we have to know about Entropy which is measure of a system's thermal energy per unit temperature that is unavailable for doing useful work. Entropy is a thermal inertia of the system. Only changes in entropy of a system can be calculated. It is a point function and is an extensive property of the system. Entropy is constant in a reversible adiabatic process. It is defined as

$$ds=\frac{\delta Q}{T} \text{ J/K}$$

Nernst suggested that Entropy change of a transformation between phases approaches zero when Temperature approaches zero. And this led to the Third Law of Thermodynamics. If entropy of every element in its stable state is taken to be zero, then every substance has a positive entropy which at T=0 may become zero. Entropy become zero for all perfectly ordered states of condensed matter. Third law excludes disordered materials.

To simplify the analysis of the system, the laws are reformulated to change the variables. The method Legendre differential transformations is used to to change the variable and that would yield functions that are fundamentally important in thermodynamics.

If a function of two variables f (x,y) describes the state of a system, which satisfies the equation

$$df = u\,dx + v\,dy$$

and to change the description to one involving a new function g (u, y), satisfying similar equation in terms of du and dy, then the necessary Legendre transform

g (u, y) is $g = f - ux$.

The g satisfies the equation

$$dg = -x\,du + v\,dy.$$

Now, consider a characteristic function H, called enthalpy defined as

$$H = U + PV$$

Since internal energy, U; pressure, P; volume, V are all state functions; H is also a state function.

In differential form,

$$dH = V\,dP + TdS,$$

where H is a function characterized by P and S.

The first law can be written as,

$$dU = TdS - PdV$$

The requirement for a characteristic function other than enthalpy was met by defining the Helmholtz free energy and is given by

$$A = U – TS$$

In differential form,

$$dA = - SdT – PdV$$

where A is a function of T and V.

Gibbs function G, is generated by a Legendre transformation of

$$dH = TdS + VdP$$

$$\text{i.e., } G = H - TS$$

which is also a state function.

In differential form

$$dG = VdP – SdT$$

where G is a function characterized by P and T

Characteristic functions U (V, S), H (P, S), A (V, T) and G (P, T) are known as thermodynamic potential functions. When the characteristic function is minimum the system will be in stable equilibrium. The thermodynamic potential is defined as the function which is minimized subject to all the constraints that are imposed on the system.

The relations among the above functions can be represented with the help of mathematical aids. If a relation exists among three variables x, y and z, then if we express z as a function of x and y; then

$$dz = (\frac{\partial z}{\partial x})_y dx + (\frac{\partial z}{\partial y})_x dy .$$

Let $M = (\frac{\partial z}{\partial x})$ and $N = (\frac{\partial z}{\partial y})_x$

Then dz= Mdx + Ndy,

where z, M and N are all functions of x and y.

Partially differentiating M with respect to y and N with respect to x,

we get $\left(\frac{\partial M}{\partial y}\right)_x = \frac{\partial^2 z}{\partial x \partial y}$

and $\left(\frac{\partial N}{\partial x}\right)_y = \frac{\partial^2 z}{\partial y \partial x}$

Since the second derivatives of right hand terms are equal it follows that

$$\left(\frac{\partial M}{\partial y}\right)_x = \left(\frac{\partial N}{\partial x}\right)_y$$

This is known as the condition for exact differentials and it applies to all for thermodynamic potentials.

Applying the above result to the our exact differentials; dU, dH ,dA,dG

We obtain;

dU = TdS – PdV ; hence $\left(\frac{\partial T}{\partial V}\right)_S = =-\left(\frac{\partial P}{\partial S}\right)_V$

dH = V dP+ TdS ; hence $\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P$

dA = - SdT – PdV ; hence $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

dG = VdP – SdT ; hence $\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P$

The four equations on the right are known as Maxwell's relations. These equations expresses relations which hold at any equilibrium state of a hydrostatic system. These equations can be used to find equivalent terms related to entropy and can provide relationships between measurable quantities and those which either cannot be measured or difficult to measure.

1.4.3 Phase Transitions

The phase is defined as a state of matter that is uniform throughout, not only in chemical composition but also in physical state. A phase transition is a change in state in rom one phase to another. It occurs when there is an abrupt change in one or more properties of the system. The electrical resistivity of a material goes from zero to a finite value in a superconducting to normal phase transition. In a ferromagnetic phase transition the magnetic properties of a system change abruptly from those of a paramagnet to those of a ferromagnet. An abrupt change in properties is one sign of a phase transition. Another sign of a phase transition is the appearance of two phases coexisting side by side.

The phases having higher thermodynamic potential will be unstable and will eventually decays into the stable one. In order to simplify the notation, we describe all thermodynamic potentials as a simple mathematical function, $\Phi(\o)$, where ø is the quantity which can vary. Thus, $\Phi(\o)$, could be the generalized Gibbs free energy as a function of volume, so that $\o = V$; or $\Phi(\o)$ could represent the magnetic free energy, F with ø the magnetization.

The thermodynamic potential is analogous to the potential energy, $V(x)$, of a particle in a one-dimensional well . Just like the particle lowers its energy by sitting at the bottom of the potential well; the thermodynamic system lowers its free energy by sitting at the bottom of the thermodynamic potential. If $\Phi(\o)$, has two minima the more stable state is the one with the lower energy. The other minimum is unstable and will eventually decay into the lower minimum. We say that the system is metastable.

One minimum could correspond to the liquid phase of matter, the other minimum to the gas phase. Suppose the two minima evolve as the temperature (or pressure) increases so that one minimum is lower over one range of temperatures, the other is stable over another range. When the system evolves from one stable minimum to the other the phase changes, say from liquid to gas

Phase transitions are of two types: First order Phase Transitions are discontinuous. They involve a latent heat. Discontinuous phase transitions are characterized by a discontinuous change in entropy at a fixed temperature. Examples are solid-liquid and liquid-gas transitions at temperatures below the critical temperature. Second order phase transitions are continuous transitions which involve a continuous change in entropy which means there is no latent heat. n. If all the first derivatives of the thermodynamic potential are continuous at the transition, we call it a continuous transition.

The entropy is continuous at a continuous transition. Examples are liquid-gas transitions at temperature above the critical temperature. Bose condensation and paramagnetic to ferromagnetic phase transitions are also examples of a second order phase transition

Common examples of phase transitions are the ice melting and the water boiling, or the transformation of graphite into diamond at high pressures. The first order phase transitions are accompanied by abrupt changes in the specific volume and entropy. A first-order phase transition is determined by the relations:

$$T1 = T2 \text{ and } P1 = P2$$

$$G1(P, T) = G2(P, T)$$

where T, P, G are the temperature, pressure, and the Gibbs thermodynamic potential, respectively. Using the above relation, we can derive the Clausius–Clapeyron equation, which defines the slope of the phase equilibrium curves:

$$\frac{dP}{dT} = \frac{\Delta S}{\Delta V}$$

where ΔS and ΔV are the volume and entropy changes at the phase transition. The second-order phase transitions include transitions associated with an emergence of magnetism, superconductivity, superfluidity, orientational order, etc. Ehrenfest proposed equations relating the slope of the phase transition curve to discontinuities in the heat capacity, compressibility, and thermal expansion coefficient.

1.5 BLACK HOLE

A black hole is a region of spacetime where gravity is so strong that no particles or even electromagnetic radiation such as light can escape from it. The story of the black hole begins with Schwarzschild's discovery of the Schwarzschild solution in 1916, soon after Einstein's foundation of the general theory of relativity.

The Schwarzchild radius is the boundary of the black hole which is determined by Karl Schwarzchild and it completely depends on the mass of Black hole. If escape velocity is greater than velocity c the light cannot escape and we have a black hole .Any object with a physical radius smaller than Schwarzchild radius will be a Black hole. Anything that crosses the event horizon needs to be travelling at speed greater than velocity of light c .We see a black sphere reflecting nothing. So it is the event horizon which is the Black part.'Hole' part in black hole comes from the Singularity.

When a massive star has exhausted the internal thermonuclear fuels in its core at the end of its life, the core becomes unstable and gravitationally collapses inward upon itself, and the star's outer layers are blown away. The crushing weight of constituent matter falling in from all sides compresses the dying star to a point of zero volume and infinite density called the singularity. This singularity is covered by Event Horizon. Radius of the sphere representing the event horizon is called the Schwarzschild radius, $Rs = \frac{2GM}{c^2}$

1.5.1 No hair theorem

According to no-hair theorem only three parameters are required to define the most general black hole. They are mass M, charge Q and angular momentum J. Black holes have no hair whereas Star has many hairs (or parameters)

1.5.2 Classes of Black hole

Based on no-hair theorem the black holes can be characterized into three,

a. Static black holes with no charge, described by Schwarzschild solution.

b. Black holes with electrical charge described by Reissner Nordström solutions

c. Rotating black holes described by Kerr solutions

a. Schwarzschild black hole

Karl Schwarzschild in 1916 gives the First solution of Einstein's equations of General Relativity. He describes gravitational field in empty space around a nonrotating mass space-time interval in Schwarzschild's solution Schwarzschild metric is a spherically symmetric black hole. It is the simplest kind parametrized by a single parameter mass, M. Its line element is defined as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\,(d\theta^2 + \sin^2\theta\; d\emptyset^2)$$

It exhibits a singularity at the Schwarzschild radius r= 2M. This is the surface below which one can no longer escape from the black hole.

b. Reissner-Nordstrom black hole

The Reissner-Nordström geometry describes the geometry of empty space surrounding a charged black hole. The German aeronautical engineer Reissner and the Finnish physicist Nordström independently solved the Einstein- Maxwell field equations for charged spherically symmetric systems, in 1916 and 1918, respectively since most stars, and thus most black holes formed from the collapse of stars, have angular momentum, it is desirable to generalize the spherical, non-rotating Schwarzschild solution to that of rotating source. So, the difference from Schwarzschild metric is that this has an additional Coulomb field. The line element for Reissner- Nordström black holes is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\emptyset^2\right)$$

As the charged black holes in a realistic environment will quickly attract opposite charges from the surroundings and get neutralized, this solution is not of astrophysical interest.

c. Kerr black hole

Both the Schwarzschild and Reissner- Nordström black holes are spinless. The solution for a rotating black hole was put forward by Kerr in 1963 with an additional 37 parameter, the angular momentum, J. The line element for black hole having mass and angular momentum is given by,

$$ds^2 = \frac{\Delta}{\rho^2} (dT - h \, sin^2\theta \, d\phi)^2 - \frac{\rho^2}{\Delta} dR^2 - \rho^2 d0^2 - \frac{sin^2\theta}{\rho^2} [(R^2 + h^2) \, d\phi - h \, dT]^2$$

where h $\equiv \frac{J}{M}$= angular momentum per unit mass,

$$\Delta = R^2 - 2GMR + h^2$$

$$\rho^2 = R^2 + h^2 \, cos^2\theta.$$

As charged black holes are not considered physically, astrophysical black holes are mainly Kerr.

d. Kerr Newman black hole

The Kerr–Newman metric is an asymptotically flat, stationary solution of the Einstein–Maxwell equations in general relativity. It describes the spacetime geometry in the region surrounding an electrically charged, rotating mass. It generalizes the Kerr metric by taking into account the field energy of an electromagnetic field, in addition to describing rotation

Such solutions do not include any electric charges other than that associated with the gravitational field, and are thus termed as vacuum solutions[1]. Newman combined the RN solution with Kerr solution and generated the spacetime geometry for a charged spinning mass. The metric equation for charged rotating black holes is as same as equation but with Δ defined as

$$\Delta = R^2 - 2GMR + h^2 + GQ^2$$

Chapter 2

BLACKHOLE THERMODYNAMICS

2.1 INTRODUCTION

Bekenstein and Hawking showed that the black holes have an entropy which is proportional to the area of the black hole. This was analogous to the second law of thermodynamics.

The entropy of black hole is given by,

$S = \frac{kC^3A}{4G\hbar}$ Bekenstein–Hawking formula

Where A is the area of the event horizon, L_P is the Planck length, G is the Newton's gravity constant, $\hbar$ is the reduced Planck's constant.

This laid a milestone in the study of black holes. In later years the four complete laws of black hole thermodynamics were introduced. These laws have a strong resemblance with the laws of thermodynamics. Thus, it became clear that the black holes do indeed behave as thermodynamic system. The crucial step in this realization was Hawking's remarkable discovery of 1974 that quantum processes allow a black hole to emit thermal flux particles.

2.2 LAWS OF BLACKHOLE THERMODYNAMICS

a) Zeroth law

By the zeroth law of black hole mechanics, the surface gravity of a stationary black hole must be constant over the event horizon of the black hole. This is analogous to the zeroth law of thermodynamics which states that the temperature is unform throughout a system in thermal equilibrium. Here the surface gravity κ

playing the role of temperature. The black holes have a well-defined temperature, which is as a matter of fact proportional to the surface gravity:

$$T = \frac{\hbar}{2\pi}\kappa$$

b) First Law

First law for a stationary black hole gives relation between change in mass M, angular momentum J and area A,

$$dM = \frac{kdA}{8\pi G} + \Omega dJ$$

where Ω is the angular velocity of the event horizon.

For a rotating charged black hole, the First law takes the form,

$$dM = \frac{kdA}{8\pi G} + \Omega dJ$$

This is analogous to the first law of ordinary thermodynamics.

According to first law of thermodynamics all the thermodynamic processes are subjected to the principle of conservation of energy. The first law states that the change in internal energy is equal to the difference of change in heat transfer and work done by the system

$$\delta E = T\delta S + work\ done.$$

We see that the analogous quantities are, E ↔ M, T ↔ ακ, and S ↔ A/8πα, where α is a constant.

c) Second Law

The Area theorem of general relativity states that the area of a black hole can never decreases in any process i.e.,

$$\Delta A \geq 0.$$

Bekenstein observed that this is analogous to the second law of thermodynamics. By second law the total entropy of a closed system can never decrease through any process. This law requires black hole to have entropy. If it carried no entropy, falling of mass into a black hole would violate the second law. But this law is not informative in its original form. For example, if an ordinary system falls into a black hole, the ordinary entropy becomes invisible to an exterior observer, so from the observer's point of view, the concept of saying increase in ordinary entropy doesn't provide any insight. Thus, the ordinary second law is transcended.

Including the black hole entropy, gives a more useful law, the generalized second law of thermodynamics the sum of ordinary entropy outside black holes and the total black hole entropy never decreases and typically increases as a consequence of generic transformations of the black hole. When matter entropy flows into a black hole, the law requires an increase in black hole entropy more than compensate of ordinary entropy from sight. During the process of Hawking radiation, the black hole's area decreases, in violation of the area theorem. The generalized second law predicts that the emergent Hawking radiation entropy shall more than compensate for the drop in black hole entropy.

d) Third Law

We have already seen the statement of third law in ordinary thermodynamics. They are:

• The entropy of a system at absolute zero temperature either vanishes or becomes independent of the intensive thermodynamic parameters.

• To bring a system to absolute zero temperature involves an infinite number of processes or steps.

The third law of black hole mechanics states that it is not possible to form a black hole with vanishing surface gravity. That is, $\kappa = 0$ cannot be achieved. A black hole with $T=0$ has, $\kappa = 0$. This corresponds to an extreme Kerr black hole with $J = M^2$

Chapter 3

ANTI DE SITTER SPACE

The most ideal black holes are those which extend to asymptotically flat empty space. But they are of little relevance. The asymptotically flat black holes cannot reach thermodynamic stability, due to the inevitable so-called Hawking radiation. In order to obtain a better understanding of the thermodynamic properties and phase transition of black holes, we must ensure that the black hole can achieve stability in the sense of thermodynamics . During the formulations of General Theory, the universe was assumed to be static. The cosmological constant was introduced by Einstein in 1917 to allow for the possibility of a static universe but was dismissed as a mistake after the expansion of the universe was discovered . Einstein considered it to be his “greatest mistake,” but it has turned out to be a required ingredient of modern cosmology since 1998 . The physical effect this constant is to impose, on a large scale, a small repulsive force . This force, if adjusted in just the right way, can be made to compensate precisely for the average gravitational attraction between the galaxies . Weinberg showed as in order to permit our existence as observers the value of cosmological constant must be very small and positive.

The black holes with zero cosmological constant describe homogenous and isotropic universe and are called flat-Minkowski space. We can also have solutions to Einstein equations with non-zero cosmological constants. If $\Lambda > 0$ the solutions tend asymptotically to de Sitter space and if $\Lambda < 0$ the solution would tend to Anti-de Sitter space. The Einstein equation with cosmological constant can be written in the form

$$R_{ab} - \Lambda g_{ab} = \frac{8\pi G}{C^4}(Tab-\frac{1}{2}g_{ab}R)$$

Just like the black holes in a flat space, the one in de Sitter and Anti de Sitter do exhibits properties including temperature, entropy and free energy. It was found that a black hole in de Sitter space would emit particles with temperature determined by the surface gravity of the black hole horizon

Anti-de Sitter space has been regarded as of little physical interest. The major reason for it is because of the negative value of Λ. It when interpreted as a negative energy density, there would be a negative energy density corresponding to it.. AdS space has no natural temperature associated with it just like the flat space. The main interest for these spaces has come from string theory and M-Theory, but also cosmological models with extra dimensions use properties of the anti-de Sitter spacetimes . A black hole in anti-de Sitter space has a minimum temperature which occurs when its size is of the order of the characteristic radius of the anti-de Sitter space .

If we take the asymptotically flat black hole as a thermodynamic system, it does not meet the requirements to achieve thermodynamic stability due to its negative heat capacity. Comparing with the asymptotically flat black holes, AdS black holes can be in thermodynamic equilibrium and stable state, because the heat capacity of the system is positive when the system parameters take certain values. Recently, increasing attention has been paid to the possibility that the cosmological constant Λ could be an independent thermodynamic parameter (pressure), and the first law of thermodynamics of AdS black hole may also be established with P-V terms In anti-de Sitter space the gravitational potential relative to any origin increases at large spatial distances from the origin. This means that the locally measured temperature of a thermal state decreases and that the total energy of the thermal radiation is finite [3].

The metric of anti-de Sitter space in static form is given by

$ds^2 = -(r)\, dt^2 + f(r)^{-1}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\emptyset^2)$

where $f(r) = 1 + \frac{r^2}{l^2}$

where L is the AdS length scale and is related to the cosmological constant as

$L^2 = \frac{-3}{\Lambda}$

Chapter:4

TAUB-NUT SPACE

4.1 INTRODUCTION

The Taub–NUT metric is an exact solution to Einstein's equations. It may be considered a first attempt in finding the metric of a spinning black hole. Taub–NUT solution was first discovered by Taub (1951), but expressed in a coordinate system which only covers the time-dependent part of what is now considered as the complete space-time.

It was initially constructed on the assumption of the existence of a four-dimensional group of isometries so that it could be interpreted as a possible vacuum homogeneous cosmological model.This solution was subsequently rediscovered by Newman, Tamburino and Unti (1963) whose initials constitute the “NUT” of the TN spacetimes.

They rediscovered it as a simple generalisation of the Schwarzschild space-time.Although they presented it with an emphasis on the exterior stationary region, they expressed it in terms of coordinates which cover both stationary and time-dependent regions.

In addition to a Schwarzschild-like parameter m which is interpreted as the mass of the source, it contained two additional parameters – a continuous parameter l which is now known as the NUT parameter, and the discrete 2-space curvature parameter which is denoted here by ϵ.In 1963, when Roy Kerr introduced the Kerr metric for rotating BHs, he came up with a 4-parameter solution, one of which was the mass of the central body and the other was its angular momentum.

The NUT-parameter or the so-called NUT charge was one of the two remaining parameters, which was eliminated from his solution because Kerr believed that it was nonphysical since it made the metric non-asymptotically flat. However, 3 other researchers interpret it either as a gravomagnetic monopole parameter of the central mass or a twisting property of the surrounding spacetime.

It is only the case in which $\in = +1$, which includes the Schwarzschild solution, that was obtained by Taub. The cases with other values of $\in$ are generalisations of the other A-metrics. We will follow the usual convention of referring to the case in which $\in = +1$ as the Taub–NUT solution.

Two different interpretations: Both of these have unsatisfactory aspects in terms of their global physical properties. In one interpretation, the space-time contains a semi-infinite line singularity, part of which is surrounded by a region that contains closed timelike curves. The other interpretation, which is due to Misner (1963), contains no singularities. These are removed only at the expense of introducing a periodic time coordinate throughout the stationary region. However, this also has other undesirable features, such that Misner was led to conclude that the complete space-time has no reasonable physical interpretation

4.2 NUT Parameter

In recent years, it has become common to refer to the NUT parameter l as the magnetic mass or the gravitomagnetic monopole moment. This interpretation is based on an analogue of one aspect of a property of one of the possible interpretations, but is not relevant in the context of the alternative global interpretation of the space-time.

In Misner's interpretation, the Taub–NUT solution has a number of properties that are usually considered to be undesirable in any reasonable representation of a spacetime. This is so much the case that Misner (1967) has presented it as "a counter-example to almost anything"

Chapter 5

Thermodynamics of Reissner Nordstrom Anti De Sitter Black hole

The RN-AdS black hole gives a spherically symmetric, stationery black hole.

It line element is determined by two parameters: charge Q and mass M.

It also contains a negative cosmological constant.

The metric equation for RN-anti de Sitter black hole is given by

$ds^2 = g_{ik}dx^i dx^k$ --------------------1.1

$= -(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2\, d\Omega^2$ ------------------------1.2

where, $f(r) = 1-\frac{2M}{r}+\frac{r^2}{L^2}+\frac{Q^2}{r^2}$ -----------1.3

$d\Omega^2$ represents the line element of unit-2 sphere, or

$d\Omega^2 = d\theta^2 + \sin^2\theta d\emptyset^2$ ---------------1.4

and L is the AdS length scale and is related to the cosmological constant as

$\Lambda =- \frac{3}{L^2}$ ----------------1.5

This black hole solution has two event horizons and one cosmological horizon .

The vacuum pressure of the AdS spacetime depends on the cosmological constant and is given by

$P = -\frac{\Lambda}{8\pi}= \frac{3}{8\pi L^2}$ -----------------1.6

f(r) satisfies the relation f(r) =0, with r, the event horizon

Solving for M,

$$M = \frac{r}{2} + \frac{Q^2}{2r} + \frac{r^3}{L^2} \quad \text{---------------1.7}$$

We have the relation between r and S

$$S = \pi r^2 \quad \text{------------------1.8}$$

Replacing r in terms of S gives

$$M = \frac{1}{2}\sqrt{\frac{S}{\pi}} + \frac{Q^2}{2}\sqrt{\frac{\pi}{S}} + \left(\frac{S}{\pi}\right)^{3/2}\frac{1}{2L^2} \quad \text{----------------1.9}$$

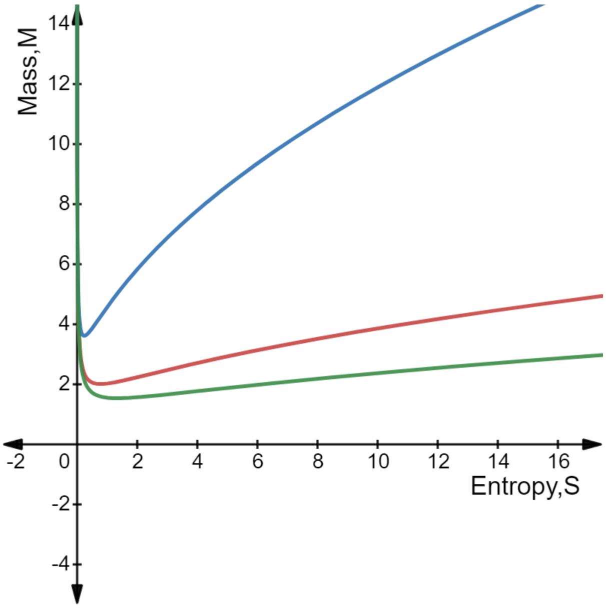


FIGURE1.1 The graph shows the variation of mass with entropy with charge Q unity and length scale L = 0.5, L = 1.0, L = 1.5

Thermodynamic Potential of a charge black hole is given by,

$$\varnothing = \frac{\partial M}{\partial Q}$$

$$= \sqrt{\frac{\pi}{S}}Q \quad \text{----------------------1.10}$$

The temperature of the black hole is given by,

$T = \frac{\partial M}{\partial S}$

$= \frac{\partial}{\partial S}\left(\frac{1}{2}\sqrt{\frac{S}{\pi}} + \frac{Q^2}{2}\sqrt{\frac{\pi}{S}} + \left(\frac{S}{\pi}\right)^{3/2}\frac{1}{2L^2}\right)$

$$\boldsymbol{T = \frac{1}{4\sqrt{\pi S}} - \frac{Q^2\sqrt{\pi}}{4^3\sqrt{S}} + \frac{3\sqrt{S}}{4L^2\, 3\sqrt{\pi}}} \text{------------------1.11}$$

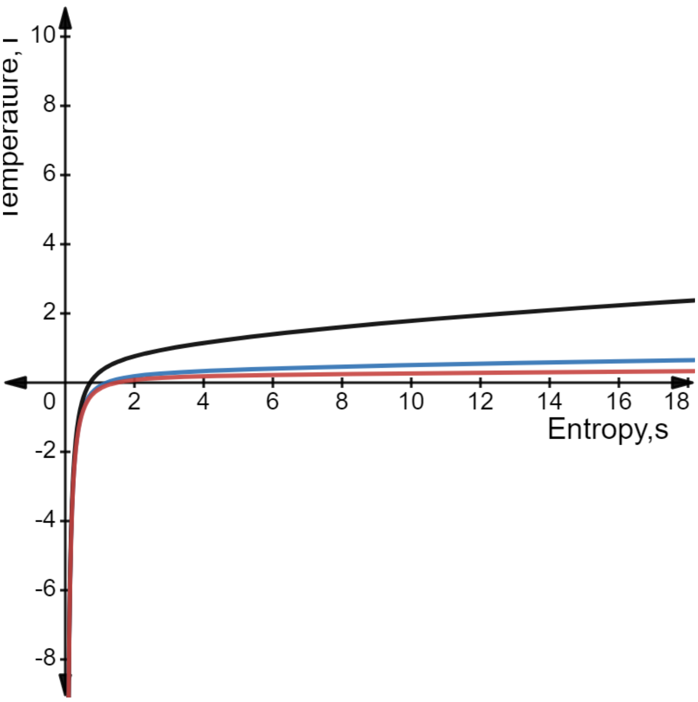


Figure 1.2 The variation of temperature with S, with L=0.5, L=1, L=1.5 and Q= 1 is shown in the graph above.

The Heat capacity is given by

$C = T\frac{\partial S}{\partial T}$

$= T\left(\frac{\partial T}{\partial S}\right)^{-1}$

$\frac{\partial T}{\partial S} = -\frac{1}{4\sqrt{\pi}\, 2S^{3/2}} + \frac{Q^2\sqrt{\pi}}{4}\frac{3}{2}S^{-5/2} + \frac{3}{4\, 2\sqrt{S\, \pi^{3/2}}L^2}$

$= \frac{1}{8L^2\pi^{3/2}S^{5/2}}\ (-L^2\pi S + L^2Q^2 3\pi^{5/2} + 3S^2)$

$$C = \left(\frac{1}{4\sqrt{\pi S}} - \frac{Q^2\sqrt{\pi}}{4^3\sqrt{S}} + \frac{3}{4L^2\,3\sqrt{\pi}}\sqrt{S}\right)\left(\frac{8L^2\pi^{3/2}S^{5/2}}{-L^2\pi S + L^2Q^2 3\pi^{\frac{5}{2}} + 3S^2}\right)$$

$$\mathbf{C} = \frac{6S^3 - 2SQ^2L^2\pi^2 + 2L^2\pi S^2}{3S^2 - L^2\pi S + 3L^2Q^2\pi^{5/2}} \quad \text{------------------1.12}$$

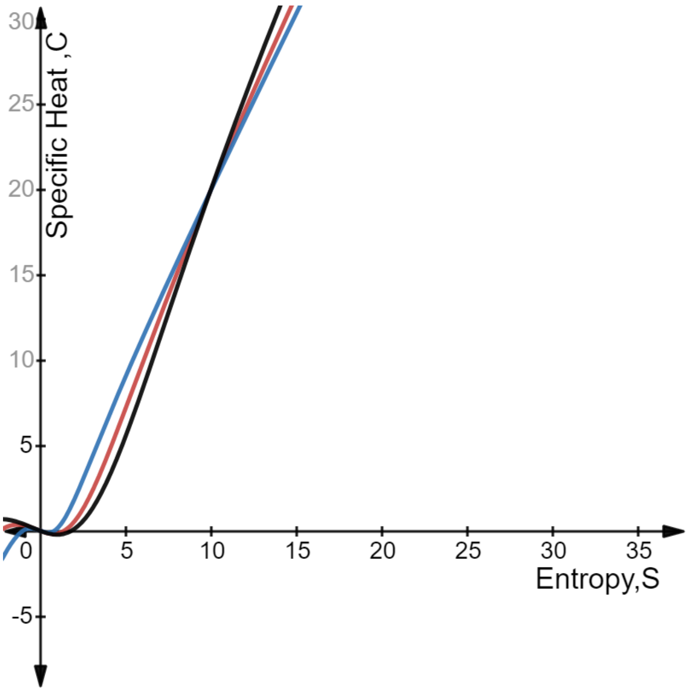


Figure 1.3 The figure shows the variation of heat capacity with entropy. The charge of the black hole is assumed to be unity and the length scale is varied as L= 0.5, L= 1.0 and L= 1.5

The free energy of the black hole given by,

F = M – TS

$$= \left(\frac{1}{2}\sqrt{\frac{S}{\pi}} + \frac{Q^2}{2}\sqrt{\frac{\pi}{S}} + \left(\frac{S}{\pi}\right)^{3/2}\frac{1}{2L^2}\right) - S\left(\frac{1}{4\sqrt{\pi S}} - \frac{Q^2\sqrt{\pi}}{4\;3\sqrt{S}} + \frac{3\sqrt{S}}{4L^2\,3\sqrt{\pi}}\right)$$

$$=\frac{1}{2}\sqrt{\frac{S}{\pi}} + \frac{Q^2}{2}\sqrt{\frac{\pi}{S}} + \left(\frac{S}{\pi}\right)^{3/2}\frac{1}{2L^2} - \frac{1}{4}\sqrt{\frac{S}{\pi}} + \frac{Q^2}{4}\sqrt{\frac{\pi}{S}} - \frac{3}{4L^2}\sqrt[3]{\frac{S}{\pi}}$$

$$\mathbf{F} = \frac{1}{4}\sqrt{\frac{S}{\pi}} - \frac{1}{4L^2}\left(\frac{S}{\pi}\right)^{3/2} + \frac{3Q^2}{4}\sqrt{\frac{\pi}{S}} \quad \text{---------------------1.13}$$

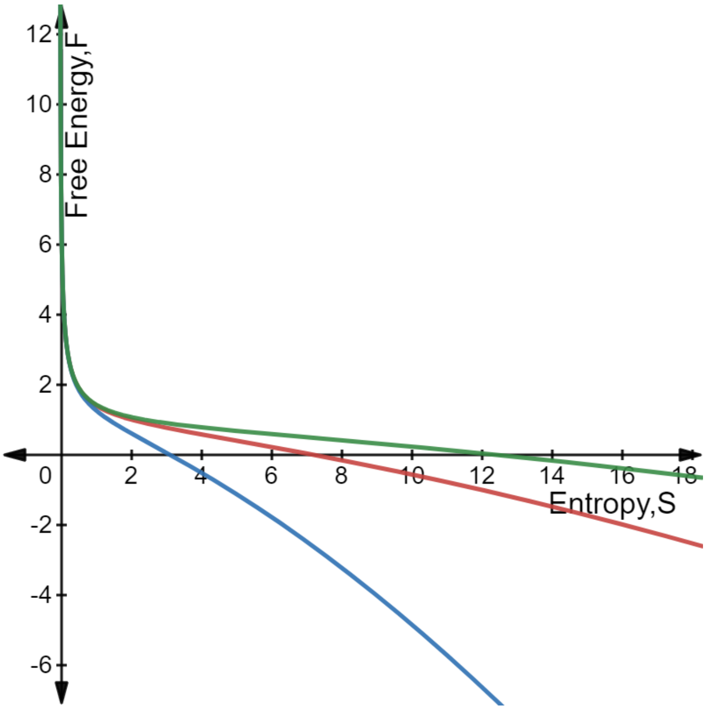


Figure 1.4 The graph showing variation of free energy for three different values of L. Q is chosen to be unity.

Chapter 6

THERMODYNAMICS OF BARDEEN BLACKHOLE

Bardeen's solution of Einstein's equation in the presence of nonlinear electromagnetic field is parametrized by mass M and charge q. The static and spherically symmetric line element is given by

$$\boldsymbol{ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + sin^2\theta d\varphi^2)} \quad (2.1)$$

where,

$$f(r) = 1 - \frac{2Mr^2}{(r^2+q^2)^{\frac{3}{2}}} \quad (2.2)$$

Using the area law, ($S = 4\pi r^2$), we can write the mass in terms of the entropy S and the q as:

$$\mathbf{M} = \frac{(\mathbf{S}+\boldsymbol{\pi}\mathbf{q}^2)^{\frac{3}{2}}}{2\sqrt{\boldsymbol{\pi}}\mathbf{S}} \quad (2.3)$$

Temperature of the black hole is given by the relation, T = $\frac{\partial M}{\partial S}$ as,

$$T = \frac{\partial}{\partial S}\left[\frac{(S+\pi q^2)^{\frac{3}{2}}}{2\sqrt{\pi}S}\right]$$

$$= \frac{3(S+\pi q^2)^{\frac{1}{2}}}{4\sqrt{\pi}S} - \frac{(S+\pi q^2)^{\frac{3}{2}}}{2\sqrt{\pi}S}$$

$$= \frac{(S+\pi q^2)^{\frac{1}{2}}}{4\sqrt{\pi}S^2}(3S - 2(S+\pi q^2)$$

$$\mathbf{T} = \frac{(\mathbf{S}-2\boldsymbol{\pi}\mathbf{q}^2)\ \sqrt{\mathbf{S}+\boldsymbol{\pi}\mathbf{q}^2}}{4\sqrt{\boldsymbol{\pi}}\mathbf{S}^2} \quad (2.4)$$

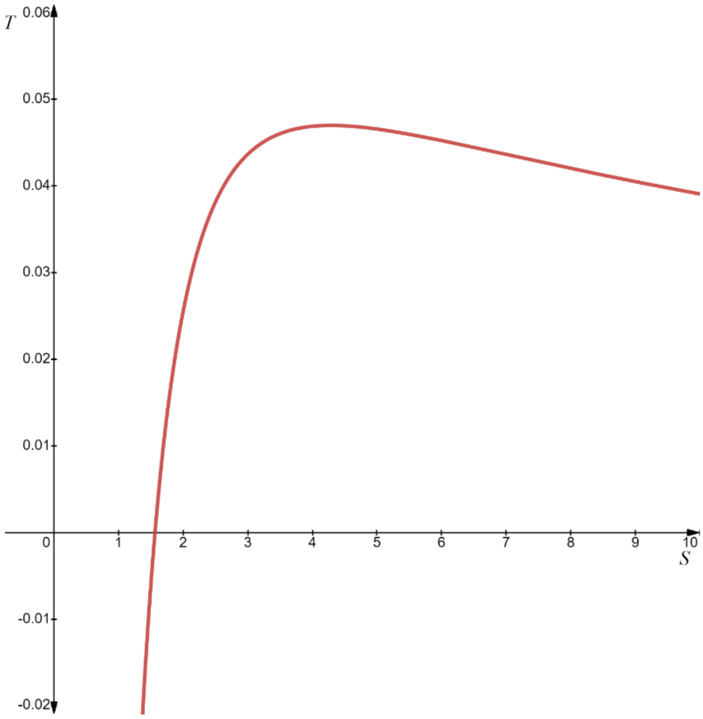


Figure 1. Variation of temperature with respect to entropy S of Bardeen black hole.

We will now calculate the heat capacity, $C = T\frac{\partial S}{\partial T}$, of the black hole and is given by

$$C = T\left[\frac{\partial T}{\partial S}\right]^{-1} = T\frac{\partial}{\partial S}\left[\frac{(S-2\pi q^2)\ \sqrt{S+\pi q^2}}{4\sqrt{\pi}s^2}\right]^{-1}$$

$$= T\ \frac{8\sqrt{\pi}S^3}{[S(S+\pi q^2)^{-\frac{1}{2}}(S-2\pi q^2)-\sqrt{S+\pi q^2}(2S-8\pi q^2)]}$$

$$= \frac{2S\ (S-2\pi q^2)\ \sqrt{S+\pi q^2}}{[S(S+\pi q^2)^{-\frac{1}{2}}(S-2\pi q^2)-\sqrt{S+\pi q^2}(2S-8\pi q^2)]}$$

Dividing numerator and denominator with $\sqrt{(S+\pi q^2)}$

$$\mathbf{C} = \frac{-2S\,(2\pi q^2 - S)\,(S+\pi q^2)}{8\pi^2 q^4 + 4\pi q^2 S - S^2} \tag{2.5}$$

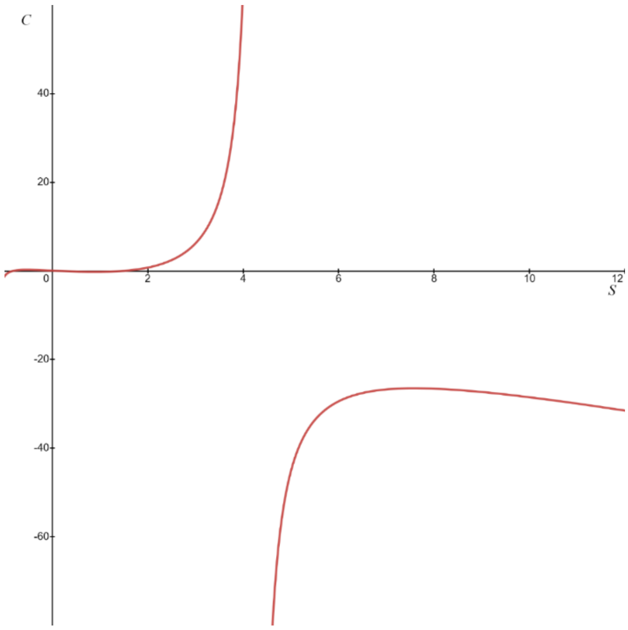


Figure 3. Variation of heat capacity with respect to entropy S of Bardeen black hole.

Then the Gibb's free energy, $F = M - TS$, is given by

$$\boldsymbol{F} = \frac{(\boldsymbol{S} + \boldsymbol{\pi q^2})^{\frac{3}{2}}}{2\sqrt{\boldsymbol{\pi S}}} - \frac{(\boldsymbol{S} - 2\boldsymbol{\pi q^2})\ \sqrt{\boldsymbol{S} + \boldsymbol{\pi q^2}}}{4\sqrt{\boldsymbol{\pi S}}} \tag{2.6}$$

Now we study whether this black hole will undergo a second order phase transition or not. This can be done in two ways.

The first method is by studying the variation of the heat capacity with entropy and we can see a discontinuity in heat capacity (Fig3) for a particular value of entropy(S = 4.1, q = 0.5). And we also note that heat capacity possesses a positive phase below this value of S and a negative phase above this value of S. The second method is by studying the variation of free energy(F) with temperature(T).

A second order phase transition is obvious for two reasons. First, of course the heat capacity shows an infinite discontinuity (at S = 4.1, where q = 0.5) and possesses both positive and negative phases. The positive phase exists for small values of S and the black hole is stable only in this region. The same result can also be seen from the parametric plot between the free energy and the temperature (Fig4) which shows a cusp type double point (at T = 0.046, where S = 4.1 and q = 0.5). The F-T variation shows that there are two branches of the curve, for one of the branches, the free energy decreases with the increase of Hawking temperature to the minimum limit, while for the other branch F increases rapidly with T. This behavior also signals a second order phase transition. (The numerical values are obtained from the corresponding graphs.)

Chapter 7

Thermodynamics of Taub-NUT Charged Blackhole

Introduction

Taub-NUT black hole solution depends on three parameters: Mass M, NUT parameter l,

$$ds^2 = -f(r)(dt - 2l\cos\theta d\varphi)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta d\varphi^2)$$

where $\boldsymbol{f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2}}$

When $l = 0$ and $m \neq 0$, the metric reduces to the *Schwarzschild solution* in which m is the familiar parameter representing the mass of the source.

Newman, Tamburino and Unti (1963) have shown that, when l is small, the inclusion of this parameter induces a small additional advance in the perihelion of approximately elliptic orbits in the stationary region of the space-time. However, when $l \neq 0$, the space-time has very different global properties to that of Schwarzschild**.**

TAUB NUT CHARGED BLACK HOLE

Charged TNBH solution depends on three parameters: Mass M, NUT parameter l, and charge q and the metric of CTNBH describes the vacuum spacetime around a source

$$\boldsymbol{ds^2 = f(r)(dt - 2lcos\theta d\varphi)^2 - \frac{dr^2}{f(r)} - (r^2 + l^2)(d\theta^2 + sin^2\theta d\varphi^2)}$$

where $\boldsymbol{f(r) = 1 - \frac{2(Mr+l^2)+q^2}{r^2+l^2}}$

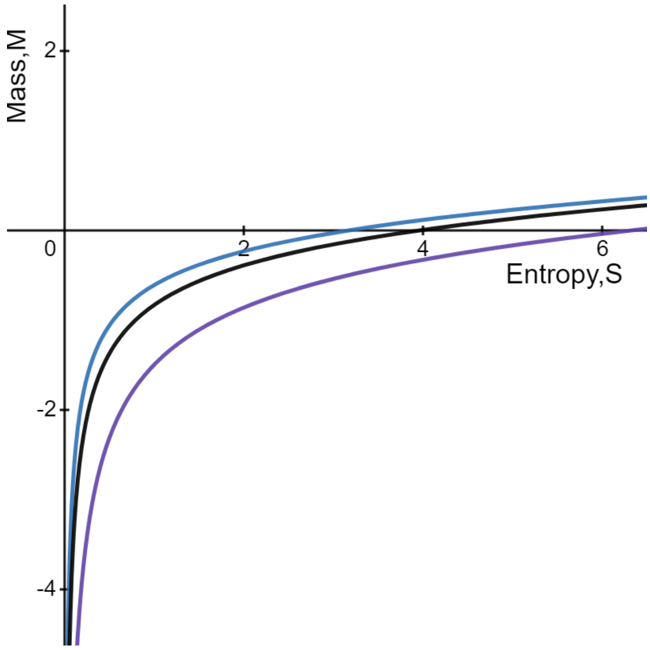
$\boldsymbol{f(r)}$ satisfies the relation $\boldsymbol{f(r) = 0}$,

Solving for M; using the relation between r and S as S=πr^2

And replacing r in terms of entropy S gives,

$$\boldsymbol{M = \frac{1}{2}\sqrt{\frac{s}{\pi}} - \frac{q^2}{2}\sqrt{\frac{\pi}{s}} - \frac{l^2}{2}\sqrt{\frac{\pi}{s}}}$$

Variation of mass with respect to entropy for length L=0,L=0.5,L=1 with charge Q=1.



The temperature of the blackhole is given by

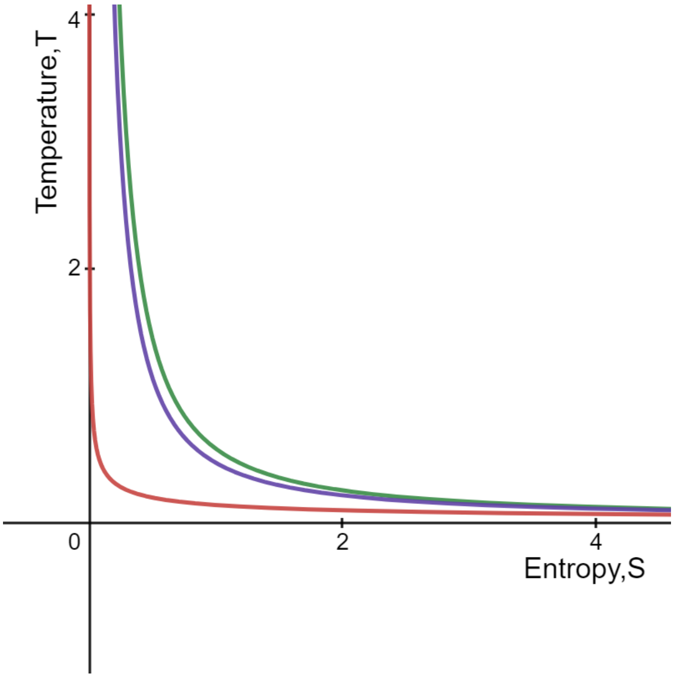
$$T=\frac{\partial M}{\partial S} = \frac{\partial}{\partial S}\left(\frac{1}{2}\sqrt{\frac{s}{\pi}} - \frac{q^2}{2}\sqrt{\frac{\pi}{s}} - \frac{l^2}{2}\sqrt{\frac{\pi}{s}}\right)$$

$$T = \frac{1}{4\sqrt{\pi s}} + \frac{q^2\sqrt{\pi}}{4s^{\frac{3}{2}}} + \frac{\sqrt{\pi} l^2}{4s^{\frac{3}{2}}}$$

$$T = \frac{\pi l^2 + s + \pi q^2}{4\sqrt{\pi}\, s^{\frac{3}{2}}}$$

$$\mathbf{T} = \frac{\mathbf{1}}{\mathbf{4\sqrt{\pi s}}} + \frac{\mathbf{q^2\sqrt{\pi} + \sqrt{\pi} l^2}}{\mathbf{4s^{\frac{3}{2}}}}$$

Variation of temperature with respect to entropy for length L=0,L=0.5,L=1 with charge Q=1



Heat capacity is given by,

$$C = T\frac{\partial S}{\partial T}$$

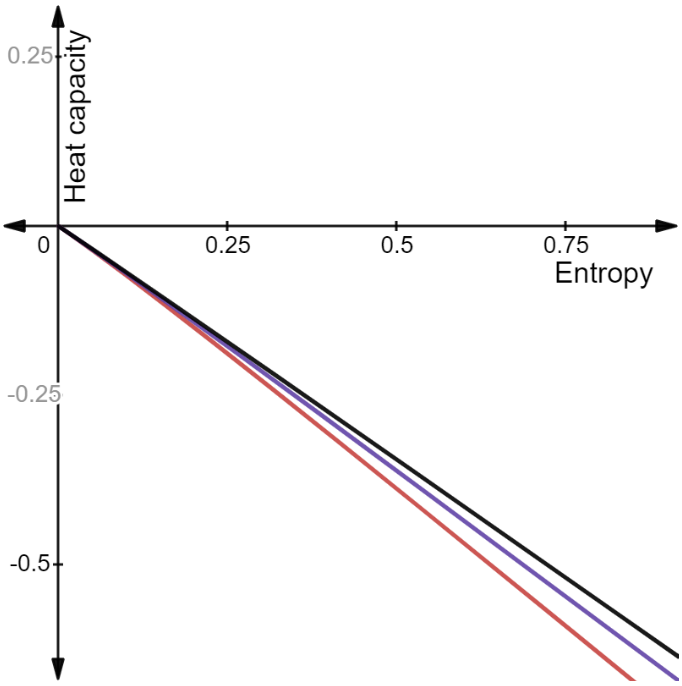
$$C = T\left[\frac{\partial T}{\partial S}\right]^{-1}$$

$$\frac{\partial T}{\partial S} = \frac{-3q^2\pi - 3l^2\pi - s}{8\sqrt{\pi}s^{\frac{5}{2}}}$$

$$C = \left(\frac{\mathbf{1}}{\mathbf{4}\sqrt{\boldsymbol{\pi s}}} + \frac{\boldsymbol{q}^{\mathbf{2}}\sqrt{\boldsymbol{\pi}}}{\mathbf{4}\boldsymbol{s}^{\frac{3}{2}}} + \frac{\sqrt{\boldsymbol{\pi}\boldsymbol{l}^{\mathbf{2}}}}{\mathbf{4}\boldsymbol{s}^{\frac{3}{2}}}\right)\left(\frac{8\sqrt{\pi}s^{\frac{5}{2}}}{-3q^2\pi - 3l^2\pi - s}\right)$$

$$\boldsymbol{C} = \left(\frac{\boldsymbol{S}(\boldsymbol{q}^{\mathbf{2}}\boldsymbol{\pi} + \boldsymbol{l}^{\mathbf{2}}\boldsymbol{\pi} + \sqrt{\boldsymbol{S}})}{-\mathbf{3}\boldsymbol{q}^{\mathbf{2}}\boldsymbol{\pi} - \mathbf{3}\boldsymbol{l}^{\mathbf{2}}\boldsymbol{\pi} - \boldsymbol{S}}\right)$$

Variation of heat capacity with respect to entropy for length L=0,L=1,L=2 with charge Q=1



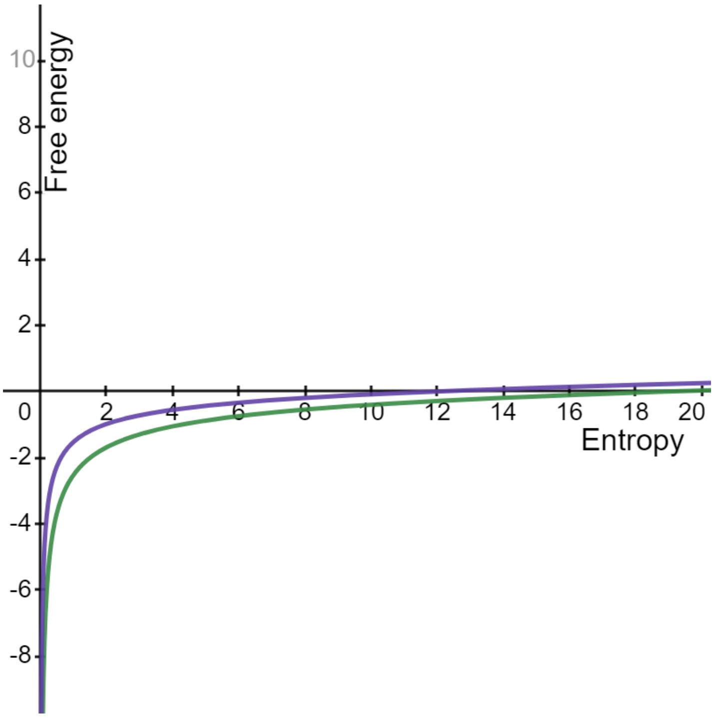
Free energy of a blackhole is given by

$$F = M - TS$$

$$F = \left(\frac{1}{2}\sqrt{\frac{s}{\pi}} - \frac{q^2}{2}\sqrt{\frac{\pi}{s}} - \frac{l^2}{2}\sqrt{\frac{\pi}{s}}\right) - S\left(\frac{1}{4\sqrt{\pi s}} + \frac{q^2\sqrt{\pi}}{4s^{\frac{3}{2}}} + \frac{\sqrt{\pi}l^2}{4s^{\frac{3}{2}}}\right)$$

$$F = \left(\frac{1}{4}\sqrt{\frac{s}{\pi}} - \frac{q^2}{4}\sqrt{\frac{\pi}{s}} - \frac{l^2}{4}\sqrt{\frac{\pi}{s}}\right)$$

The graph shows the variation of free energy with entropy with length scale L=1 and L=0.5 with charge Q=1


Free energy
Entropy
10
8
6
4
2
0
-2
-4
-6
-8
2
4
6
8
10
12
14
16
18
20

Chapter -8

CONCLUSION

In this project we studied RN-AdS black hole, Bardeen black hole and Taub NUT black hole. We have derived the thermodynamic quantities, and plotted their variations with respect to entropy.

In Bardeen black hole, from the temperature entropy diagram we have eliminated the possibility of a first order phase transition. A second order phase transition is obvious, since the heat capacity shows an infinite discontinuity and possesses both positive and negative phases. The positive phase exists for small values of S and the black hole is stable only in this region.

In RN-AdS black hole, below a certain value of entropy the mass increases with decrease in entropy and the mass and gives an infinite discontinuity at zero entropy. At very low values of entropy the temperature of the black hole shows an abnormal behaviour. For RN-AdS black hole the heat capacity is negative lower values of entropy and reaches zero at a certain value. The free energy of the black holes falls to negative above a certain value.

For Taub-NUT black hole, there is a smooth variation of mass with entropy. The black hole shows finite temperature for all values of entropy. For Taub-NUT, heat capacity is negative and it indicates that the black hole is unstable and the free energy falls to negative above a certain value.

CHAPTER-9

BIBLIOGRAPHY

1. Weinberg, Steven. Gravitation and Cosmology: Principles and applications of general theory of relativity, Wiley India Pvt. Limited, 2008

2. Narlikar, Jayant V. An introduction to relativity. Cambridge University Press, 2010

3. Hall, Michael J W. General Relativity: An Introduction to Black Holes, Gravitational Waves, and Cosmology. N.p., Morgan & Claypool Publishers, 2018.

4. Guidry, Mike. Modern General Relativity: Black Holes, Gravitational Waves, and Cosmology. United Kingdom, Cambridge University Press, 2019.

5. Ong, Yen Chin. Evolution of Black Holes in Anti-de Sitter Spacetime and the Firewall Controversy. Germany, Springer Berlin Heidelberg, 2015.

6. Britannica, The Editors of Encyclopaedia. "Black hole". Encyclopedia Britannica, 13 Oct. 2020, https://www.britannica.com/science/black-hole. Accessed 25 March 2021.

7. Puri, S. P.. General Theory of Relativity. India, Pearson Education India, 2013.

8. Bowley, Roger, and Sánchez, Mariana. Introductory Statistical Mechanics. United Kingdom, Clarendon Press, 1999

9. Kumar, Prasanna. Thermodynamics. India, Pearson Education India, 2013.

10.Poisson, Eric. A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics. Cambridge University Press, 2004.

11. Hawking, S. W. "Black holes and thermodynamics." Phys. Rev. D. 13,2, Jan 1976, 191-197

12. Jacob d, Bekenstein (2008) Bekenstein-hawking Entropy, Scholarpedia, 3(10): 7375. http://www.scholarpedia.org/article/Bekenstein-Hawking_entropy Accessed on (11-04-2021)

13.Grøn, Ǿyvind, Sigbjørn Hervik. Einstein's General Theory of Relativity. Springer-Verlag New York, 2007

14. Zemansky, Mark W, and Richard Dittman. Heat and Thermodynamics: An Intermediate Textbook. New York: McGraw-Hill, 1997.

15. https://en.wikipedia.org/wiki/Kerr%E2%80%93Newman_metric

16. Stenflo, J. (2019). Origin of the cosmological constant. Astrophysics and Space Science. 364.

17. Einstein, Albert, and Robert W. Lawson. Relativity: The Special and General Theory. New York: Holt, 1921. Print. 4

18. S. W. Hawking and Don N. Page. Thermodynamics of black holes in antide sitter space. Communications in Mathematical Physics, 87(4):577–588, Dec 1983

19. Al-Badawi, Kanzi, and Sakallı, "Fermionic and Bosonic Greybody Factors as Well as Quasinormal Modes for Charged Taub NUT Black Holes."

20. Jerry.B.Griffith, Podolsky, "Exact space-times in general relativity".