Project Report

On

A STUDY ON MATHEMATICAL MODELING OF NUMERICAL WEATHER FORECASTING

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CERTIFICATE

This is to certify that the dissertation entitled, **A STUDY ON MATHEMATICAL MODELING OF NUMERICAL WEATHER FORECASTING** is a bonafide record of the work done by Ms. **PRIYA FRANCIS** under my guidance as partial fulfillment of the award of the degree of **Master of Science in Mathematics** at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of **NISHA OOMMEN**, Assistant Professor, Department of Mathematics, St. Teresa's College(Autonomous), Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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Chapter 1

Introduction and History

1.1 History

The history of numerical weather prediction considers how current weather conditions as input into mathematical models of the atmosphere and oceans to predict the weather and future sea state has different over the years. The roots of numerical weather prediction can be traced back to the work of Vilhelm Bjerknes, a Norwegian physicist who has been called the father of modern meteorology. In 1904, he published a paper suggesting that it would be possible to forecast the weather by solving a system of nonlinear partial differential equations. A British mathematician named Lewis Fry Richardson spent three years developing Bjerknes's techniques and procedures to solve these equations. In 1920s, it was not until the advent of the computer and computer simulation that computation time was reduced to less than the forecast period itself. ENIAC was used to create the first forecasts via computer in 1950, and over the years more powerful computers have been used to increase the size of original datasets as well as include more difficult versions of the equations of motion. The development of global forecasting models led to the first climate models. The development of limited area models facilitated developments in forecasting the tracks of tropical cyclone as well as air quality in the 1970s and 1980s. Because the output of forecast models based on atmospheric dynamics needs corrections near ground level, model output statistics [MOS] were developed in 1970s and 1980s for individual forecast points. The MOS apply statistical techniques to post-process the output of dynamical models with the most recent surface observations and the forecast points climatology. This technique can correct for model resolution as well as model biases. Even with the increasing power of supercomputers, the forecast skill of numerical weather models only extends to about two weeks into the future. Since the density and quality of observations together with the chaotic nature of the partial differential equations used to calculate the forecast introduce errors which double every five days. The use of model ensemble forecasts since the 1990s helps to define the forecast uncertainty and extend weather forecasting farther into the future than otherwise possible.

1.2 Introduction

Weather forecasting can be defined as the act of predicting future weather conditions or an effort to indicate the weather conditions which are expected to occur. Weather forecasting is the application of Science and Technology to predict the state of the atmosphere for a future time and a given location. Human beings have attempted to predict the weather informally for times, and formally since at least the nineteenth century. Weather forecasts are made by collecting qualitative data about the present state of the atmosphere and using scientific understanding of atmospheric processes to project how the atmosphere will change within the next few hours. Once, an all-human endeavour based mainly upon changes in barometric pressure, current weather conditions and sky conditions, forecast models are now used to fix future conditions.

The dynamics of the atmosphere is governed by physical, chemical, and even Biological processes which are commonly described by systems of time and space dependent nonlinear partial differential equations. Since this kind of mathematical description is slightly complicated, the form of the equations' exact solution is usually unknown. In order to explore its properties or to compute its approximation, further Mathematical methods are needed. The resulting atmospheric models are then to forecast the weather situation, the concentration of an air pollutant, or even the changes in climate.

There are two well-known NWP models namely, National Weather Service's Global Forecast System (GFS) and the European Centre for Medium-Range Weather Forecast, known as ECMWF model. These models are also known as the American and European Models, respectively. It is generally mentioned at some context that European models has produced most accurate global weather forecasts.

Chapter 2

Numerical Weather Prediction

Numerical weather prediction uses mathematical models of the atmosphere and oceans to predict the weather based on current weather conditions. Post processing techniques such as model output statistics have been developed to improve handling of errors in numerical predictions.

Both the significance of weather forecasts and the need of knowing more about atmospheric developments were understood equally soon after the first wind warnings were published. But very few people realised that mathematics could be used to describe these processes and produce more exact forecasts than synoptic meteorology ever could. In the early 20th century, scientists, in particular vilhelm Bjerknes and Lewis Fry Richardson, established numerical weather forecasting, which is based on applying physical laws to the atmosphere and solving mathematical equations associated to these laws. The discovery of chaos theory and not least the development of computers greatly improved the quality of forecasts. Today, meteorologists constantly improve the various forecasting models designed by the world's leading weather services.

2.1 Numerical Weather Prediction Equations

The Primitive Equations are used as the forecast equations in NWP models. Vilhelm Bjerknes first recognized that numerical weather prediction was possible in principle in 1904. He proposed that weather prediction could be seen as an initial value problem in mathematics. Since equations direct how meteorological variables change with time, if we know the initial condition of the atmosphere, we can solve the equations to obtain new values of those variables at a future time (i.e., make a forecast). To represent an NWP model in its simplest form,we can write:

$$\frac{\Delta A}{\Delta t} = F(A) \tag{2.1}$$

Where ΔA gives the change in a forecast variable at a particular point in space.Δt gives the change in time(how far into the future we are forecasting), F(A) represents terms that can cause changes in the value of A. This equation means that the change in forecast variable A during the time period t is equal to the cumulative effects of all processes that force A to change. Future values of meteorological variables are solved for by finding their initial values and then adding the physical forcing that acts on the variables over the time period of the forecast. This is stated as

$$A^{forecast} = A^{initial} + F(A)\Delta t \tag{2.2}$$

where F(A) stands for the combination of all of the kinds of forcing that can occur.

2.2 Primitive Equations

Primitive Equations are used as the forecast equations in NWP models.

2.2.1 Momentum Equations

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv \tag{2.3}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu \tag{2.4}$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{2.5}$$

Where u is the Zonal velocity – velocity in the east-west direction tangent to the sphere, v is the meridional velocity – velocity in the north -south direction tangent to the sphere, w is the vertical velocity, ρ is the density, p is the pressure, f is the Coriolis force, g is the acceleration of gravity.

2.2.2 Thermodynamic Equation

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - w\frac{\partial T}{\partial z} + T \tag{2.6}$$

Where T is the temperature.

2.2.3 Mass Continuity Equation

$$\frac{\partial P}{\partial t} = -u\frac{\partial P}{\partial x} - v\frac{\partial P}{\partial y} - w\frac{\partial P}{\partial z} - \rho\Delta V \tag{2.7}$$

Where p is the pressure, ρ is the density

2.2.4 Ideal gas law

$$P = \rho RT \tag{2.8}$$

where P is the pressure,ρ is the density, R is the gas constant, T is the temperature.

2.2.5 Hydrostatic eqaution

$$P_1 = P_0 e^{-\frac{gz_1}{RT}} \tag{2.9}$$

Chapter 3

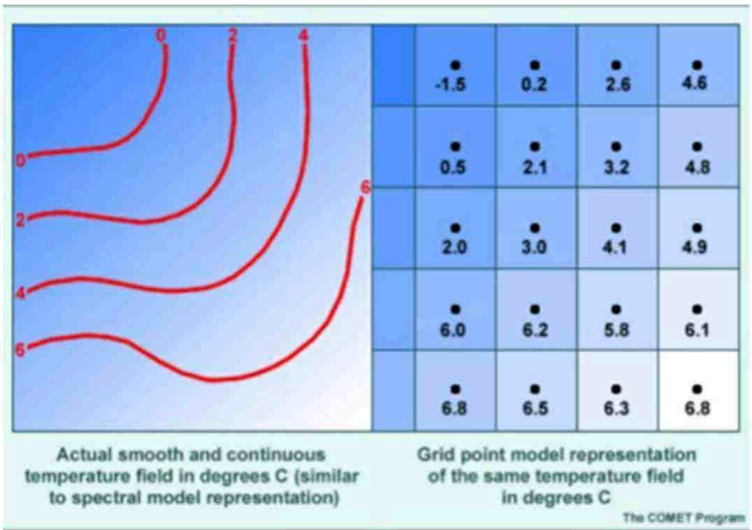
Numerical Weather prediction models

An Atmospheric model is a mathematical model made around the full set of Primitive Equations which rule atmospheric motions. Most atmospheric models are numerical. i.e. they are Equations of motion. The horizontal domain of a model is either Global, covering the entire Earth or Regional(limited area), covering only part of the Earth.

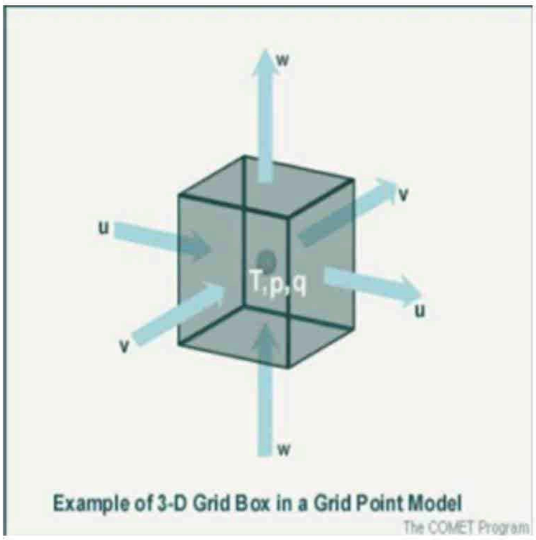
The different types of models run are:

3.1 Grid Point Model

In the real atmosphere, wind, pressure, temperature, and moisture differ from location to location in a smooth, constant way. Grid point models, however, make their calculations on a fixed collection of spatially disconnected grid points. The values at the grid points actually represent an area average over a grid box. The continuous temperature field shown in the next graphic, therefore, must be represented at each grid point as shown by the black numbers in the right panel of the previous graphic. The temperature value at the grid point represents the grid box volume average.



Grid point models really represent the atmosphere in three-dimensional grid cubes, such as the one shown below. The temperature, pressure, and moisture (T, p, and q), shown in the centre of the cube, represent the average conditions throughout the cube. The east-west winds (u) and the north-south winds (v), located at the sides of the cube, represent the average of the wind components between the centre of this cube and the centre of the adjacent cubes. Similarly, the vertical motion (w) is represented on the upper and lower faces of the cube. This procedure of variables within and around the grid cube (called a staggered grid) has advantages when calculating derivatives. It is also physically spontaneous, average thermodynamic properties inside the grid cube are represented at the centre, whereas the winds on the faces are associated with changes into and out of the cube.



Grid point models must use difference methods to solve the forecast equations. In the real atmosphere, advection regularly occurs at very small scales. The greater the distance between grid points, the less likely the model will be able to detect small-scale variations in the temperature and moisture fields. The lack of resolution introduces errors into the solution of the finite difference equation. Shortages in the ability of the finite difference approximations to calculate gradients and higher order derivatives exactly are called Truncation Errors.

3.1.1 Shapes of Grids

Richardson's effort of predicting weather using grid points set thestage for future development of grids in different shapes. In order to accommodate thespherical shape of the earth and represent the equations more exactly and efficiently,there are different grid shapes used in numerical models.

Rectangular / Square Grids

The rectangular or square grids is the most commonly used grids in the NWP models. The rectangular grid is simple in nature but suffers from the polar problem where the lines of equal longitude known as meridians, converge to points at the poles. The poles are unique points and may cause violations of global conservation laws within the model. To maintain computational stability near the poles, small integration time-steps could be used, but at great expense. The high resolution in the east-west direction near the poles would be wasted because the model uses lower resolution.

A rotated grid can overcome the polar problem for limited area models, but forglobal models, other grid shapes are used. For example, Kurihara proposed to use 'skipped'or 'Kurihara' grid. Unfortunately use of the Kurihara grid causes fakehigh pressure to develop at the poles. As a result, their use has been strictly limited orabandoned in finite difference models. However, problems due to the use of the Kuriharagrid can be resolved by using more accurate numerical schemes. In the late 60's and early70's, the application of quasi-uniform grids

was proposed as a method to avoid the polarproblem of the grid-point models. For example, the Global Forecasting System (GFS)model has roughly a square grid near the equator, a more rectangular grid in the mid-latitudes,and a triangular grid near the poles, eventually converging to a point at the poles.Another example of a model that uses the rectangular grid type is the North American MesoscaleModel (NAM). When compared to the resolution of the GFS, the NAM does not havea grid stretching problem since the model calculates variables close to the poles. This is dueto the NAM not depend on a latitude-longitude system for creating its grid bounds and acceptingto a more precise horizontal measurement system. The other problem with the latitude-longitude grid is the need for special filters to deal with the pole singularities. They also do not scale well on massively parallel computers.

TRIANGULAR GRIDS

Triangular grids are not used as often in models as are rectangular grids. One form of quasiuniformgrid whose base element is a triangle is the spherical geodesic grid. Icosahedralgrids, first introduced in the 1960s, give almost homogeneous and quasi-isotropic coverageof the sphere. The grid is made by dividing the triangular faces of an icosahedron into smallertriangles, the vertices of which are the grid points. Each point on the face or edge of one ofthe faces of the icosahedron is surrounded by six triangles making each point the centre of ahexagon. The triangular faces of the icosahedrons are arranged into pairs to form rhombuses,five around the South Pole and five around the North Pole. The poles are chosen as twopentagonal points where the five rhombuses meet. The main advantage of the geodesic gridis that all the grid cells are nearly the same size. The uniform cell size allows for computational stability even with finite volume schemes.

HEXAGONAL GRIDS

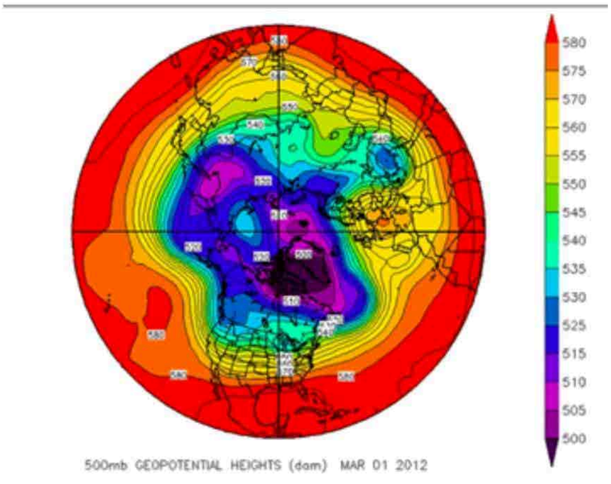
Similar to triangular grids, hexagonal grids are also not used as often as the rectangular/square grids. In this method, variables are calculated at each grid intersection between differenthexagons, in addition to be-

ing calculated in the centre of the hexagonal grid.Sadourneydescribes in detail how the spherical icosahedral-hexagonal grid is constructed. They solved the non-divergent barotropic vorticity equation with finite differencemethods on the icosahedral-hexagonal grids. Majewski et al. develops an approachthat uses local basis functions that are orthogonal and conform perfectly to the spherical surface.A study done by Thuburnshows a method of creating a global hexagonal grid, butthen using a finite differencing method to calculate the rate-of-change of different variableswithout having to create triangles within the hexagonal grids. The space differencingscheme using the icosahedral-hexagonal grid gives a satisfactory approximation to the analyticalequations given an initial condition and remains nonlinearly stable, for any condition.

Thuburnalso noted that his method may not be as exact as those which included anadditional point in the centre of the hexagonal grid, but his method was computationallyfaster, and was able to accurately show the polar regions since there was no need of stretchingthe grid in that region. The other advantages of the hexagonal grid are: (i) Removes thepolar problem. (ii) Permits larger explicit time steps. (iii) Most isotropic compared to othergrid types. (iv) Conservation of quantities in finite volume formulation. Can be generalized easily to arbitrary grid structures.

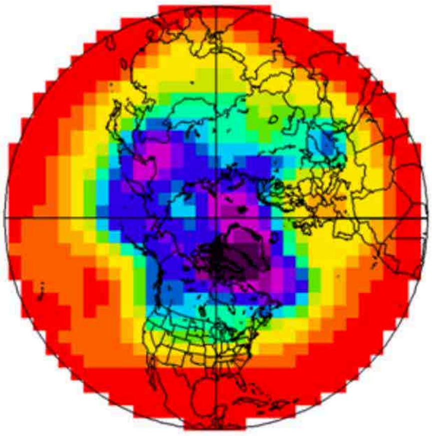
3.2 Spectral Model

Spectral models represent the spatial variations of Meteorological variables as a finite series of waves of Differing wavelengths Consider the map below that shows the 500 – mb height pattern for March 1, 2012. From this polar stereographic perspective, we can see many long waves located around the globe.



Some variables, such as this 500-mb height pattern, are not well-represented by a grid-point model. Instead, a model that incorporates wave patterns would be better suited.

Now we can see a gridded version of this height data

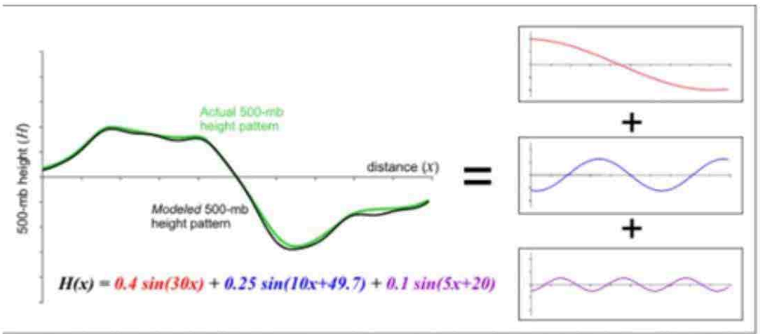


One of the first things is that we have almost completely lost the sense of the waves that make up the pattern. Furthermore, by gridding the data in such a way, some of the better features of the height patterns have vanished. Since many variables in the atmosphere can be pictured by wave-like structures [rather than square boxes]. It turns out there is a numerical weather prediction model that uses waves instead of grid boxes, a SPECTRAL MODEL.

The primary spectral model used by the National Centres for Environmental Prediction is the Global Forecast System, or GFS, for short. Rather than dividing the atmosphere into a series of grid boxes, the GFS describes the present and future conditions of the atmosphere by solving mathematical equations whose graphical solutions consist of a series of waves.

The important concept of the Spectral modelling lies in the idea that any wavelike function can be replicated by adding various basic waves together.

Let's see an example, The green line as an actual 500-mb height line, for example, stretch across the U.S. and represent a long-wave point and trough. A spectral model first approximates this pattern by adding together a set of simple wave functions in this example, variations of a trigonometric function called the "sine" are used. We were able to closely duplicate the green curve by adding together three different wave functions (the red, blue, and purple curves). The resulting black curve is fairly close to the green curve and has a simple equation (mathematically speaking, of course) that a computer has no difficulty interpreting.



A spectral model attempts to replicate wavelike patterns in key atmospheric variables (here, a sample 500-mb height line represented by the green curve) by adding together simple wave functions (red, blue, and purple curves). The resulting sum (black curve) usually represents the observed pattern fairly well, depending on the number of simple waves contained in the sum. The model then uses the relatively simple mathematical equation describing the sum in its computations.

Thus, the first step in using a spectral model is to analyse the present patterns in the observed atmospheric variables and then closely replicate these patterns using sums of simple wave functions. One advantage to this approach is that the way in which wave functions change in space and time is well known. This mathematical fact helps toprove a major advantage of spectral models that they run faster on computers. Given these computational time savings, spectral models better give themselves to longer-range forecasts than grid-point models like the NAM [North American Mesoscale Model]. Grid-point models push modern supercomputers to their limits just to mix out a respectable threeor four-day forecast. However, the GFS is routinely run out to 384 hours (16 days) four times a day (starting at 00 UTC, 06 UTC, 12 UTC and

18 UTC). One final advantage of spectral models is that their solutions are available for every point on the globe, rather than being tied to a regular grid collection.

3.3 Hydrostatic Model

Most grid point models and all spectral models in the current operational NWP models are hydrostatic. This means that no vertical accelerations are calculated clearly. The hydrostatic assumption is valid for synoptic- and global-scale systems and for some mesoscale phenomena. An important exception is deep convection, where resistance becomes an important force. Hydrostatic models account for the effects of convection using statistical parameterizations approximating the larger-scale changes in temperature and moisture caused by non-hydrostatic processes.

The main advantageof hydrostatic models, it can run fast over limited area domains, providing forecasts in time for operational use.

3.4 Non-Hydrostatic Model

Currently, most non-hydrostatic models are grid point models. They are generally used in forecast or research problems requiring very high horizontal resolution (from tens of meters to a few kilometres) and cover relatively small domains.

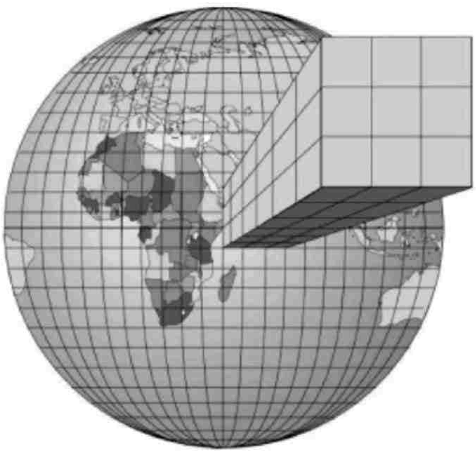
Use of the non-hydrostatic primitive equations, directly forecasting vertical motion used for forecasting small scale phenomena. Predict realistic looking, detailed mesoscale structure and consistent impact on surrounding weather, resulting in either superior local forecasts or large errors.

Chapter 4

Mathematical methods used in weather forecasting

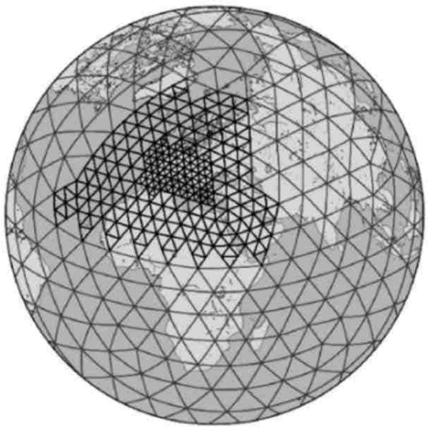
4.1 Finite Difference method

The traditional grid structure is based on dividing the Earth's surface into a large number of squares, such that there is a high air column above each square,



A rectangular grid around the Earth with air columns above the surface.

The atmosphere is then divided into a number of layers, resulting in a three-dimensional grid, in which the primitive equations can be solved for each grid point. In general, the layers are much thinner close to the Earth's surface than the layers in the upper atmosphere, as most weather events happen relatively close to the ground. Processes in the upper atmosphere influence the weather, so the whole of the atmosphere has to be considered in a forecasting model. Most models also include several hidden layers, so as to take the air and water exchange between atmosphere and ground into account. Over the years, the resolution of the grids has become higher (i.e. the edge length of each square has become smaller).



A triangular grid as used by the DWD.

The world's leading weather services such as the British Met Office and the German DWD use three different grids, a global grid crossing the whole planet, a so-called regional model covering Europe (and North America in the case of the Met Office's model), and a local model covering the UK or Germany, respectively. The regional model of the DWD has a resolution of 7 km, and the local model has a resolution of up to 2.8 km (the Met Office's models have coarser resolution as the Met Office does not work with non-hydrostatic equations). Both of these local models, and all of the Met Office models, are based on a rectangular grid, whereas the DWD's global model is based on a triangular grid with a 40 km resolution. The great advantage of the triangular grid is that the primitive equations can be solved in air parcels close to the poles without any problems, as opposed to the rectangular grid, where the longitudes approach each other, resulting in wrong calculations.

Generally, in a terrain following coordinate system, the grid-spacing in the λ-direction is given by $\Delta\lambda$, similarly $\Delta\phi$ and $\Delta\zeta$ represent the spacing in the ϕ-direction and the ζ-direction respectively. As in, λ

denotes the longitude, ϕ denotes the latitude and ζ is the height coordinate. The position of the grid points in the computational space is defined by,

$$\lambda_i = \lambda_0 + (i-1)\Delta\lambda \qquad i = 1, ..., N_\lambda \tag{4.1}$$

$$\phi_j = \phi_0 + (j-1)\Delta\phi \qquad j = 1, ..., N_\phi \tag{4.2}$$

$$\zeta_k = k \qquad k = 1, ..., N_\zeta \tag{4.3}$$

Here, N_α is the number of grid points in the α-direction, λ_0 and ϕ_0 are the values of λ and ϕ in the southwest corner of the model domain.

Now, the primitive equations have to be re-written in finite difference form. For this, we need to define approximations to the derivatives at a definite grid point x_l in terms of finite differences. The value of a variable ψ at x_l is given by ψ_l and the finite difference for ψ_l is given using the values of $\psi_l + 1$ and $\psi_l - 1$, i.e. the values at the two adjacent grid points. The behaviour of these two terms can be defined using Taylor expansion:

$$\psi_{l+1} = \psi_l + \psi_l^{'}\Delta x + \psi_l^{''}(\frac{\Delta x^2}{2!}) + \psi_l^{'''}(\frac{\Delta x^3}{3!}) \tag{4.4}$$

$$\psi_{l-1} = \psi_l - \psi_l^{'}\Delta x + \psi_l^{''}(\frac{\Delta x^2}{2!}) - \psi_l^{'''}(\frac{\Delta x^3}{3!}) \tag{4.5}$$

Subtracting the second from the first expansion gives the centred finite difference approximation to the first derivative of ψ_1

$$\psi_l^{'} = \frac{\psi_{l+1} - \psi_{l-1}}{2\Delta x} + E \tag{4.6}$$

The term,

$$E = (\frac{\Delta x^2}{3! * 2!})(2\psi_l^{''}) \tag{4.7}$$

gives the truncation error and can be omitted since it is small. However, the lowest power of the difference of x, Δx, in E gives the order

of the approximation. This scheme has order 2. As mentioned above, the higher the order, the more accurate the approximation. In general, centred finite difference approximations are better than forward or backward approximations, which can also be derived from the Taylor expansions for ψ_{l+1} and ψ_{l-1}. Solving the first expansion for ψ_l' gives the forward approximation:

$$\psi_l' = \frac{\psi_l - \psi_{l-1}}{\Delta x} + E \tag{4.8}$$

These schemes both have order 1, but there are conditions where it is favourable to use these approximations instead of a centred approximation. It is also possible to derive the finite difference approximations to the second and third derivatives of ψ_l.

Not only space, but also time has to be discretized, and time derivatives can also be denoted as finite difference approximations, that is in terms of values at distinct time levels. A time step is denoted by Δt, and a discrete time level is given by $t_n = t_0 + n\Delta t$ with t_0 being the initial time for integration. The grid point value of the variable ψ_l at time tn is denoted by ψ_l^n and its derivative can be expressed as a centred finite difference approximation:

$$(\frac{\partial \psi}{\partial t})^n = \frac{\psi_l^{n+1}}{2\Delta t} + O \tag{4.9}$$

Again, O represents the truncation error and can be omitted.

There are two different finite difference schemes, the explicit scheme and the implicit scheme. The explicit scheme is much easier to solve than the implicit one, as it is possible to compute the new value of ψ_1 at time n+1 for every grid point, provided the values of ψ_1 are known for every grid point at the current time step n. But the choice of the time step is limited in order to keep the scheme constant. The implicit scheme, on the other hand, is absolutely constant, but it results in a system of simultaneous equations, so is more difficult to solve. Both explicit and implicit schemes are used in current forecasting models.

The approximations described above are very simple examples showing the general idea of finite differences. When the primitive equations are expressed in terms of finite differences, the equations soon become very long and take some computational effort to solve. Explicit time integration can be made more effective, though, by applying a so-called mode-splitting technique. This means that the primitive equations are subdivided into forcing terms f_ψ referring to slowly varying modes and source terms s_ψ directly related to the fast-moving sound waves:

$$\frac{\partial\psi}{\partial t} = f_\psi + s_\psi \tag{4.10}$$

The terms f_ψ are integrated over big time steps Δt. These time steps are then subdivided into several small-time steps Δt, over which the terms s_ψ are integrated. In the case that $s_\psi = 0$, we get, using a $2\Delta t$ bound interval,

$$f_n^\psi = \frac{\psi^{n+1} - \psi^{n-1}}{2\Delta t} = f_\psi(\psi^{n-1}, \psi^n, \psi^{n+1}) \tag{4.11}$$

representing a set of equations that can be solved using Gaussian elimination. For equations including acoustically active terms, i.e. where acoustic and gravity waves have to be taken into account, the finite difference is given by,

$$S_\psi^m + f_\psi^m = \frac{\psi^{m+1} - \psi^m}{\Delta\tau} \tag{4.12}$$

The superscript m is the time step counter for the integration over the small-time steps ΔT within the bound interval used above.

The term f_ψ^n is constant throughout the small-time step integrations, but the value of $\psi_{n+1}$is not known before the last one of these integrations has been completed. Therefore, the finite difference for f^{n_ψ} has to be re-written as,

$$f_n^\psi = \frac{\overline{\psi^{n+1}} - \psi^{n-1}}{2\Delta t} = f_\psi(\psi^{n-1}, \psi^n, \overline{\psi^{n+1}}) \tag{4.13}$$

The term $\overline{\psi^{n+1}}$ is the result of a process called averaging.We assume that the mean value of ψ^{n+1} does not vary as fast with respect to both space and time than deviations from the mean would. The notation for averaging is

$$\overline{\psi^{a\phi}} = \frac{1}{2}[\psi(\psi + a\frac{\Delta\phi}{2}) + \psi(\phi - a\frac{\Delta\phi}{2})] \tag{4.14}$$

with a being an integer. The notation is similar for the longitude ψ.

If we re-write the primitive equations using finite differences, we get a linear tridiagonal system of simultaneous equations which can be written in the general form,

$$A_k\overline{\psi}_{k-1}^{n+1} + B_k\overline{\psi}_k^{n+1} + C_k\overline{\psi}_{k+1}^{n+1} = D_k \tag{4.15}$$

The terms A_k, B_k and C_k are matrix diagonals, whereas D_k is an inhomogeneous term including the appropriate boundary conditions. The equation system can be solved for using a solving method based on Gaussian elimination and back-substitution.

4.2 The Spectral method

The spectral method was already invented in the 1950s, but it took a while before the method was applied in forecasting models. In 1976, the Australian and Canadian weather services were the first ones to accept this method, which is now used by a range of weather services across the globe.The European Forecasting Centre ECMWF adopted it in 1983. One of the advantages of the spectral method is that the primitive equations can be solved in terms of global functions rather than in terms of approximations at specific points as in the finite difference method. For the ECMWF, this is the better option as they need a global model in order to produce medium-range weather forecasts. For the spectral method, the atmosphere has to be represented in terms of spectral components. In the ECMWF model, the atmosphere is divided into 91 layers (in comparison, the DWD's and the Met Office's global models have 40 layers), with the number of layers in the boundary layer equalling the number of layers in the uppermost 45 km of the atmosphere. The partial differential equations are represented in terms of spherical harmonics, which are truncated at a total wave number of 799. This corresponds to a grid length of roughly 25 km (the DWD's and the Met Office's global model has a resolution of 40 km). While using the spectral method, we assume that an unknown variable ψ can be approximated in terms of a sum of N+1 linearly dependent basis functions $\psi_n(x)$:

$$\psi \approx \psi_N = \sum_{n=0}^{N} a_n \phi_n(x) \tag{4.16}$$

When this series is substituted into an equation of the form $L\psi = f(x)$, where L is a differential operator,you get a so-called residual function:

$$R(x : a_0, a_1, ..., a_N) = L\Psi_N - f \tag{4.17}$$

The residual function is zero when the solution of the equation above is exact, therefore the series coefficients an should be chosen such that the residual function is minimised, that it is as close to zero as possible. In the majority of cases, polynomial approximations, such as Fourier series or Chebyshev polynomials, are the best choice, but when it comes to weather forecasting, the use of spherical coordinates demands that spherical harmonics are used as expansion functions. This increases the difficulty of the problem, and the computational effort required to solve it. A simple example that can be solved in terms of a Fourier series shows the idea of the spectral method. One of the processes described by the primitive equations is advection (which is the transport of for instance heat in the atmosphere), and the non-linear advection equation is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{4.18}$$

This can be re-written in terms of the longitude λ:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \lambda} = 0 \tag{4.19}$$

Having chosen appropriate boundary conditions, the equation can be expanded in terms of a finite Fourier series:

$$u(\lambda, t) = \sum_{m=-M}^{M} u_m(t) e^{im\lambda} \tag{4.20}$$

where the u_m are the complex expansion coefficients and M is the maximum wave number. The advection equation then,

$$\sum_{m=-M}^{M} \frac{du_m}{dt} e^{im\lambda} + \sum_{m=-2M}^{2M} F_m e^{im\lambda} = 0 \tag{4.21}$$

where F_m is a series in terms of them.

As each of the terms on the left-hand side of the equation has been truncated at a different wave number, there will always be a residual function. There are several methods which convert differential equations to distinct problems, for example the least-square method or the Galerkin method, and which can be used in order to choose the time derivative such that the residual function is as close to zero as possible.

It is difficult to calculate the non-linear terms of a differential equation in the setting of the spectral method, but you can get around this problem by using a so-called transform method. Most commonly, Fast Fourier Transforms are used, but in principle all transform methods make it possible to switch between a spectral representation and a grid-point representation. Using a transform method requires three steps, which will be shown for the non-linear term,

$$u\frac{\partial u}{\partial \lambda} \tag{4.22}$$

In the advection equation above. Firstly, the individual components of the non-linear term u and

$$D = \frac{\partial u}{\partial \lambda} \tag{4.23}$$

are expressed in terms of spectral coefficients at discrete grid points λ_i,

$$u(\lambda_1) = \sum_m u_m e^{im\lambda_1} \tag{4.24}$$

$$D(\lambda_1) = \sum_m imu_m e^{im\lambda_1} \tag{4.25}$$

Secondly, the advection term, that is the product of these components,

is calculated at every grid point in the discretised space

$$F(\lambda_1) = u(\lambda_1)D(\lambda_1) \tag{4.26}$$

Then you can return to the spectral space and calculate the Fourier coefficients,

$$F_m = \frac{1}{2\pi}\sum_{1} F(\lambda_1)e^{im\lambda_1} \tag{4.27}$$

This procedure has to be done at every time level, so results in a significant amount of calculation. Furthermore, products with more than two components suffer from aliasing, meaning that waves that are too short to be resolved for a certain grid resolution falsely appear as longer waves. Still, using transform methods is necessary in order to solve differential equations in spectral space. As mentioned above, a dependent variable ψ has to be expanded in terms of spherical harmonics rather than Fourier coefficients when spherical coordinates are used. Spherical harmonics $Y_n^m(\lambda, \phi)$ are the angular part of the solution to Laplace's equation. The vertical components of velocity

$$u_z \equiv \frac{dz}{dt} \tag{4.28}$$

transform like scalars, so can be expanded in terms of spherical harmonics straightaway. It is slightly more complicated for the horizontal components

$$u_\lambda \equiv \frac{d\lambda}{dt} and u_\psi \equiv \frac{d\psi}{dt} \tag{4.29}$$

$$U = u_\lambda sin\phi = \sum_{m,n}^{\infty} u_{m,n} Y_n^m(\lambda, \phi) \tag{4.30}$$

$$V = u_\phi sin\phi = \sum_{m,n}^{\infty} v_{m,n} Y_n^m(\lambda, \phi) \tag{4.31}$$

with the spherical harmonics

$$Y_n^m(\lambda, \phi) = e^{im\lambda}P_n^m(\phi) \tag{4.32}$$

$$Y_n^{-m}(\lambda, \phi) = e^{-im\lambda} P_n^m(\phi)(-1)^m \tag{4.33}$$

where m and n are non-negative integers such that $n \geq m$. Here, m is the zonal wave number, and n is the total wave number. The term $P_n^m(\phi)$ is the associated Legendre function used in spherical harmonics (but will not be explained here). In general, spectral method algorithms are more difficult to program than their finite difference counterparts, also the domains in which they are used have to be regular in order to keep the high accuracy of this method. However, the spectral method has a number of advantages, for example, there is no pole problem when the method is used. At the poles, the solutions to differential equations become infinitely differentiable, therefore the poles are usually excluded from the spectral space, which actually simplifies the method. Furthermore, it can handle finite elements of higher orders than the finite difference method can. As a result, the solutions of many problems are very precise. The high accuracy also results in the fact that the models do not need as many grid points as in the finite difference method, and computers on which the method is run require less memory space. Summing up, the spectral method gives much more accurate results than the finite difference method. Many weather services still use the finite difference method though because it is much easier to implement.

4.3 Finite Element method

A third technique for finding approximate solutions to partial differential equations and hence to the primitive equations is the finite element method. It is quite similar to the spectral method in that a dependent variable ψ is defined over the whole domain in question, rather than at discrete grid points used in the finite difference method. Furthermore, a finite series expansion in terms of linearly independent functions approximates the variation of ψ within a specified element (e.g. a set of grid points). Unlike the spectral method, the basic functions are not globally, but only locally non-zero, also they are low-order polynomials rather than high-order polynomials. The domain for which the partial differential equations have to be solved is divided into a number of subdomains, and a different polynomial is used to approximate the solution for each subdomain. These approximations are then combined into the primitive equations. A condition for the finite element method to work, however, is that ψ is continuous between neighbouring elements. The fact that only low-order polynomials can be used is reflected in the relatively low accuracy, but the amount of necessary calculations is much smaller than for finite differences or for spectral methods. On the other hand, there are a number of choices for the basic functions, and depending on which functions are used, the finite element method can give very accurate results when it is applied to irregular grids. Thus, the use of this method is not restricted to triangular and rectangular grids only as the finite difference method. This is probably more important in engineering and fluid dynamics, where this method is most widely used. However, scientist is constantly trying to improve existing and find new mathematical methods that model atmospheric processes better than the methods in use nowadays.

Chapter 5

Applications of Atmospheric Models

5.1 Climate Modeling

A climate model is a computer program designed to simulate Earth's climate in order to understand and predict its behaviour. Climate models are mainly based on a set of mathematical equations that describe the physical laws which rule the behaviour of the atmosphere and ocean, and their interactions with other parts of Earth's climate system. (e.g. land surface and ice sheets). Processes for which fundamental equations are not known or which occur at scales smaller that the grid resolution (e.g. clouds, vegetation) are represented actually. The mathematical equations define how variables such as temperature, pressure, and wind change over time. These equations are solved using very large supercomputers. Observations are also used to develop the models, mainly in the testing phase. These observations come from tools such as ocean signals, weather balloons, satellites, and instrumented commercial aircraft. When the forcing factors (e.g. intensity of the sun, concentrations of greenhouse gases, dust from volcanic eruptions) are prescribed to the model, they can be used to simulate the past and present climates, and possible future climates given situations of future anthropogenic emissions. The models represent Earth's climate by dividing the surface, ocean and atmosphere into a grid. Imagine each part of the Earth has its own box. In global models the spacing (or size) of these boxes is typically in the range of 100-300 km. In regional climate models the spacing is typically smaller, 10-50 km. For each box, the change in a variable (such as wind, temperature, or rainfall) over a specified amount of time is calculated. The time step (the amount of time between each calculation) depends on the size of the grid boxes and is usually a few minutes to about half an hour in order to solve the equations with sufficient accuracy. Models can be made up of millions of grid boxes, and are run over many thousands of time steps.This can result in the simulations taking months to produce. A number of simulations (an ensemble) are made for each scenario to estimate the mean climate and the uncertainty due to natural climate variations. The climate system is very complex. Models allow us to test theories in a controlled environment. Climate models can help us predict how the climate might differ in the future. They can improve our understanding of variables such as temperature, precipitation, oceanic currents and sea ice cover. Climate models are the only scientific tool with the potential for making regional predictions about future climate. We are confident that models provide useful information because they are based on well-known physical laws and reproduce many features of the observed climate, including how it has changed in the past.

5.2 Air Quality Modeling

Air quality dispersion modelling uses computer simulation to predict air quality concentrations from various types of emission sources. For pollutants emitted through a stack, it considers the emission rate, stack height, stack diameter, and stack gas temperature and velocity, as well as the effect of nearby buildings and terrain. Other emission sources like vehicle traffic or wind erosion from storage loads are represented as 2-dimensional area sources or 3-dimensional volume sources. Air quality dispersion models use meteorological data such as temperature, wind direction, and wind speed to calculate concentrations. Modelling is often used to predict possible impacts on air quality from new or adapting

emission sources. Model predicted concentrations are compared with the national and state ambient air quality standards to ensure protection of Minnesota's air quality in light of potential future emissions. Modelling can also be used to site ambient air monitors and inform human health and ecological risk valuations.

Chapter 6

Conclusion

Numerical forecast is known for the accurate data as observed during the forecast at the beginning of its run or at initial conditions we can say. As it is known that weather changes rapidly from one place to another, tomorrow's weather is definitely influenced by today's weather, and similarly next week's weather can be affected by today's weather a continent away. Therefore, lots of worldwide data is required to make the predictions. Numerical Weather Prediction is imprecise because the equations used by the models to simulate the atmosphere are not accurate. It leads to some error in predictions. Moreover, as we do not receive many weather observations from mountain regions or over the oceans, therefore, many gaps persist in the initial data. And so, the computer's prediction of how that initial state will evolve will not be entirely accurate if initial conditions were not completely known.

REFERENCES

[1] **Sarah N Collins, Robert S James, Pallav Ray, Katherine Chen, Angie Lassman and James Browniee, Chapter4, Grids in Numerical weather and climate models Grid point model , ", Grids in Numerical weather and climate models", Chapter4.**

[2] **Numerical Weather Climate Prediction, A book by Thomas Tomkins Warner**

[3] `https://en.m.wikipedia.org/wiki/Atmospheric_modelAtmosphericModels`

[4] `https://www.pca.state.mn.us/air/air-quality-modeling`

[5] `https://www.rmets.org/metmatters/climate-modelling`

[6] **Numerical Weather Climate Prediction, A book by Thomas Tomkins Warner**

[7] **Model Type(Adapted from COMET Online NWP modules)**

[8] `https://mathshistory.st-andrews.ac.uk/HistTopics/Weather_forecasts/4.1`