A STUDY ON TENSOR ANALYSIS

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IN MATHEMATICS

By

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CERTIFICATE

This is to certify that the dissertationentitled "A STUDY ON TENSOR ANALYSIS"is a bonafide record of the work done by SRUTHI.S.PILLAIunder my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of ALKA. BENNY, Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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Contents

INTRODUCTION

Tensor analysis is a branch of Mathematics concerned with relations or laws that remain valid regardless of the system of co-ordinates used to specify the quantities.Tensors were invented as an extension of vectors to formalize the manipulation of geometric entities arising in the study of mathematical manifolds.

Tensor analysis or Tensor calculus which is an extension of vector calculus to tensor field was developed by Gregorio Ricci-Curbastro and his student Tullio Levi-Civita.Contrasted with the infinitesimal calculus, tensor calculus allows presentation of physics equations in a form that is independent of the choice of coordinates of the manifold.

Tensor is defined as an objective entity having components that change according to a transformation law.Tensors have many applications in geometry and Physics.Tensor calculus has many real life applications in Physics and Engineering including elasticity,continnum,electromagnetism etc.

In creating the general theory of relativity,Albert Einstein argued that the laws of physics must be the same no matter what co-ordinate system is used.This led him to express those laws in terms of the tensor equations.While tensors had been studied earlier, it was the success of Einstein's general theory of relativity that gave rise to the current widespread interest of mathematicians and physicists in tensors and their applications.

Chapter 1

INTRODUCTION TO TENSOR ANALYSIS

1.1 PRELIMINARIES

Definition 1.1 n-dimensional space*:*

An ordered set of n variables say,$x^1, x^2, ...x^n$ *is called the coordinates of a point in an n dimensional space.The set of all these points together forms an n-dimensional space,denoted by* V_n

Definition 1.2 Einsteins Summation Convention*:*

Consider the sum of the series $S = a_1x^1 + a_2x^2 + ... + a_nx^n = \sum_{i=1}^{n} a_ix^i$*. By using summation convention, the sigma sign is dropped and the convention is written as*

$\sum_{i=1}^{n} a_ix^i = a_ix^i$

This convention is called Einsteins Summation Convention.This can be stated as 'If a suffix occurs twice in a term, once in the lower position and once in the upper position then that suffix implies sum over defined range.'

Definition 1.3 Dummy index*: An index that is repeated in a given term is called a dummy suffix. It is also called Umbral or Dextral Index.*

Definition 1.4 Free index*: Any index occuring only once in a given term is called a Free index.*

Definition 1.5 Kronecker Delta*: The symbol* δ^i_j *,called Kronecker Delta is defined by*

$$\delta^i_j = \begin{cases} 1, if \ i = j \\ 0, if \ i \neq j \end{cases}$$

It is also denoted by the symbols $\delta^{ij} and \delta_{ij}$

PROPERTIES:

1.If $x^1, x^2, ..., x^n$ are independent coordinates , then

$\frac{\partial x^i}{\partial x^j} = 0, if \ i \neq j$

$\frac{\partial x^i}{\partial x^j} = 1, if \ i = j$

This implies that $\frac{\partial x^i}{\partial x^j} = \delta^i_j$

2.$\delta^i_i = n$

3.$a^{ij}\delta^j_k = a^{ik}$

4. $\delta^i_j \delta^j_k = \delta^i_k$

The superscripts are used to denote the components of a contravariant tensor. The subscripts are used to denote the components of a covariant tensor.The components of a mixed tensor is denoted by both superscripts and subscripts.

Definition 1.6 Tensor *:A Tensor is a mathematical object analogous to, but more general than a vector, which is represented by an array of components that are functions of the coordinates of the space.*

Definition 1.7 *:* Tensor Analysis *Tensor Analysis is the branch of Mathematics concerned with relations or laws that remain valid regardless of the* *system of coordinates used to specify the quantities.*

Tensor Calculus is an extension of vector calculus to tensor fields. Contrasted with the infinitesimal calculus, tensor calculus allows presentation of physics equations in a form that is independent of the choice of coordinates of the manifold.

Definition 1.8 Invariant:*A function $f(x^1, x^2, ..., x^n)$ is called scalar or an invariant if its original value does not change on the transformation of coordinates from $x^1, x^2, ..., x^n$ to $\overline{x}^1, \overline{x}^2, ...\overline{x}^n$ i.e.,*
$\phi(x^1, x^2, ..., x^n) = \overline{\phi}(\overline{x}^1, \overline{x}^2, ..., \overline{x}^n)$

Scalar is called a tensor of rank zero.

1.2 CONTRAVARIANT AND COVARIANT VECTORS

Let $(x^1, x^2, ..., x^n) or x^i$ be coordinates of a point in X-coordinate system and $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^n)$ or $\overline{x}^i$ be the coordinates of the same point in the Y-coordinate system.

Definition 1.9 *Let $A^i, i = 1, 2, ..., n$ be n functions of the coordinates $x^1, x^2, ..., x^n$ in X-coordinate system.If the quantities A^i are transformed to $\overline{A}^i$ in Y-coordinate system then according to the law of transformation*

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^j} A^j \ or \ A^j = \frac{\partial x^j}{\partial \overline{x}^i} \overline{A}^i$$

Then A^i are called components of contravariant vector.

Definition 1.10 *Let $A_i, i = 1, 2, ..., n$ be n functions of the coordinates $x^1, x^2, ..., x^n$ in the X-coordinate system. If the quantities $A_i$are transformed to $\overline{A_i}$ in the Y-coordinate system, then according to the law of transformation*

$$\overline{A_i} = \frac{\partial x^j}{\partial \overline{x}^i} A_j \ \ or\ A_j = \frac{\partial \overline{x}^i}{\partial x^j} \overline{A_i}$$

Then A_i are called components of a covariant vector.

The contravariant (or covariant) vector is also called a contravariant (or covariant) tensor of rank one.

Example 1.1 *If x^i be the coordinate of a point in n-dimensional space , show that dx^i are components of a contravariant vector.*

Solution:
Let $x^1, x^2, ..., x^n$ be coordinates in X- coordinate system and $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$ be the coordinates in the Y-coordinate system.
If

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^1} dx^1 + \frac{\partial \overline{x}^i}{\partial x^2} dx^2 + ... + \frac{\partial \overline{x}^i}{\partial x^n} dx^n$$

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^j} dx^j$$

It is the law of transformation of the contravariant vector. So, dx^i are components of a contravariant vector.

Example 1.2 *Show that $\frac{\partial \phi}{\partial x^i}$ is a covariant vector where ϕ is a scalar function.*
Solution:
Let $x^1, x^2, ..., x^n$ be the coordinates in the X- coordinate system and $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$ be the coordinates in the Y- coordinate system.

Consider $\quad \phi(\overline{x}^1, \overline{x}^2, ..., \overline{x}^n) = \phi(x^1, x^2, ..., x^n)$

$$\partial\phi = \frac{\partial\phi}{\partial x^1}\partial x^1 + \frac{\partial\phi}{\partial x^2}\partial x^2 + ... + \frac{\partial\phi}{\partial x^n}\partial x^n$$

$$\frac{\partial\phi}{\partial\overline{x}^i} = \frac{\partial\phi}{\partial x^1}\frac{\partial x^1}{\partial\overline{x}^i} + \frac{\partial\phi}{\partial x^2}\frac{\partial x^2}{\partial\overline{x}^i} + ... + \frac{\partial\phi}{\partial x^n}\frac{\partial x^n}{\partial\overline{x}^i}$$

$$\frac{\partial\phi}{\partial\overline{x}^i} = \frac{\partial\phi}{\partial x^j}\frac{\partial x^j}{\partial\overline{x}^i}$$

$$or \quad \frac{\partial\phi}{\partial\overline{x}^i} = \frac{\partial x^j}{\partial\overline{x}^i}\frac{\partial\phi}{\partial x^j}$$

It is the law of transformation of the components of a covariant vector. So, $\frac{\partial\phi}{\partial x^i}$ *is the component of a covariant vector.*

Example 1.3 *:Show that the component of the tangent vector on the curve in an n-dimensional space are components of a contravariant vector.*

Solution:

Let $\frac{dx^1}{dt}, \frac{dx^2}{dt}, ..., \frac{dx^n}{dt}$ *be the components of a tangent vector of the given point* $(x^1, x^2, ..., x^n)$ *i.e.,* $\frac{dx^i}{dt}$ *is the component of the tangent vector in the X-coordinate system. Let the component of the tangent vector of the point* $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^n)$ *in the Y- coordinate system are* $\frac{d\overline{x}^i}{dt}$.

Then $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$ *are functions of* $x^1, x^2, ..., x^n$ *which are functions of t.*

So,

$$\frac{d\overline{x}^i}{dt} = \frac{\partial\overline{x}^i}{dx^1}\frac{dx^1}{dt} + \frac{\partial\overline{x}^i}{dx^2}\frac{dx^2}{dt} + ... + \frac{\partial\overline{x}^i}{dx^n}\frac{dx^n}{dt}$$

$$\frac{d\overline{x}^i}{dt} = \frac{\partial\overline{x}^i}{dx^j}\frac{dx^j}{dt}$$

It is the law of transformation of the components of a contravariant vector. So, $\frac{dx^i}{dt}$ *is component of a contravariant vector.*

i.e., the component of tangent vector on the curve in the n-dimensional space are the components of a contravariant vector.

Example 1.4 *: In rectangular cartesian coordinates, the components of acceleration vectors are ($\ddot{x}$,$\ddot{y}$). Find these components in polar coordinates.*

Solution:

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. *This implies ,*

$$dx = dr\cos\theta - (r\sin\theta)d\theta \qquad dy = dr\sin\theta + (r\cos\theta)d\theta \tag{1.1}$$

By solving these equations we get.,

$$dr = \cos\theta dx + \sin\theta dy \qquad d\theta = (\frac{-1}{r}\sin\theta)dx + (\frac{1}{r}\cos\theta)dy \tag{1.2}$$

We put,

$$(x, y) = (x^1, x^2),\ (r, \theta) = (\overline{x}^1, \overline{x}^2) \tag{1.3}$$

$$(\ddot{x}, \ddot{y}) = (A^1, A^2),\ (a_r, a_\theta) = (\overline{A}^1, \overline{A}^2) \tag{1.4}$$

where a_r, a_θ *are components of acceleration vectors in polar coordinates.*

Now , from (1.2) using (1.3)and (1.4) we get

$$d\overline{x}^1 = (\cos\theta)dx^1 + (\sin\theta)dx^2$$

$$d\overline{x}^2 = (\tfrac{-1}{r}\sin\theta)dx^1 + (\tfrac{1}{r}\cos\theta)dx^2$$

Hence,

$$\frac{\partial\overline{x}^1}{\partial x^1} = \cos\theta,\ \frac{\partial\overline{x}^1}{\partial x^2} = \sin\theta \tag{1.5}$$

$$\frac{\partial \overline{x}^2}{\partial x^1} = -\frac{1}{r}\sin\theta, \ \frac{\partial \overline{x}^2}{\partial x^2} = \frac{1}{r}\cos\theta \tag{1.6}$$

Thus,

$$a_r = \overline{A}^1 = \frac{\partial \overline{x}^1}{\partial x^1}A^1 + \frac{\partial \overline{x}^1}{\partial x^2}A^2 = (\cos\theta)\ddot{x} + (\sin\theta)\ddot{y} \tag{1.7}$$

and

$$a_\theta = \overline{A}^2 = \frac{\partial \overline{x}^2}{\partial x^1}A^1 + \frac{\partial \overline{x}^2}{\partial x^2}A^2 = (-\frac{1}{r}\sin\theta)\ddot{x} + (\frac{1}{r}\cos\theta)\ddot{y} \tag{1.8}$$

From (1.1), we get

$$\dot{x} = \tfrac{dx}{dt} = (\cos\theta)\dot{r} - r(\sin\theta)\dot{\theta}$$

$$\ddot{x} = (\cos\theta)\ddot{r} - 2(\sin\theta)\dot{\theta}\dot{r} - r(\cos\theta)\dot{\theta}^2 - r(\sin\theta)\ddot{\theta} \tag{1.9}$$

$$\dot{y} = (\sin\theta)\dot{r} + r(\cos\theta)\dot{\theta}$$

$$\ddot{y} = (\sin\theta)\ddot{r} + 2(\cos\theta)\dot{\theta}\dot{r} - r(\sin\theta)\dot{\theta}^2 + r(\cos\theta)\ddot{\theta} \tag{1.10}$$

Using (1.9) and (1.10)., we get from (1.7) and (1.8) that

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = \ddot{\theta} + \tfrac{2}{r}\dot{r}\dot{\theta}$$

Example 1.5 *:The components of a covariant vector in rectangular cartesian systems are*

$$A_1 = \tfrac{y}{x}\ , \quad A_2 = \tfrac{x}{y}$$

Find these in polar coordinates.

Solution:

Let the corresponding components in polar coordinates be $\overline{A_1}, \overline{A_2}$. Then we can write

$$\overline{A_1} = \frac{\partial x^1}{\partial \overline{x}^1} A_1 + \frac{\partial x^2}{\partial \overline{x}^1} A_2 \tag{1.11}$$

$$\overline{A_2} = \frac{\partial x^1}{\partial \overline{x}^2} A_1 + \frac{\partial x^2}{\partial \overline{x}^2} A_2 \tag{1.12}$$

Similarly from (1.1) we get,

$$dx^1 = d\overline{x}^1 \cos\theta - (r\sin\theta) d\overline{x}^2$$

$$dx^2 = d\overline{x}^1 \sin\theta + (r\cos\theta) d\overline{x}^2$$

Hence.,we get

$$\tfrac{\partial x^1}{\partial \overline{x}^1} = \cos\theta,\ \tfrac{\partial x^1}{\partial \overline{x}^2} = -r\sin\theta$$

$$\tfrac{\partial x^2}{\partial \overline{x}^1} = \sin\theta,\ \tfrac{\partial x^2}{\partial \overline{x}^2} = r\cos\theta$$

Using these relations and (1.11) and (1.12) we get,

$$\overline{A_1} = \cos\theta(\tfrac{y}{x}) + \sin\theta(\tfrac{x}{y}) = \sin\theta + \cos\theta$$

$$\overline{A_2} = r\sin\theta(\tfrac{y}{x}) + r\cos\theta(\tfrac{x}{y}) = -r\tfrac{\sin^2\theta}{\cos\theta} + r\tfrac{\cos^2\theta}{\sin\theta}$$

1.3 CONTRAVARIANT TENSOR OF RANK TWO

Let $A^{ij}(i, j = 1, 2, ...n)$ be functions of coordinates $x^1, x^2, ..., x^n$ in X-coordinate system. If the quantities A^{ij} are transformed into $\overline{A^{ij}}$ in the Y-coordinate system having the coordinates as $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$,then according to the law of transformation

$$\overline{A}^{ij} = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial \overline{x}^j}{\partial x^l}A^{kl}$$

Then such A^{ij} are called the components of contravariant tensor of rank two.

1.4 COVARIANT TENSOR OF RANK TWO

Let$A_{ij}(i,j = 1,2,...,n)$be n^2 functions of coordinates $x^1, x^2, ..., x^n$ in X-coordinate system. If the quantities A_{ij} are transformed into $\overline{A_{ij}}$ in the Y-coordinate system having the coordinates as $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$,then according to the law of transformation

$$\overline{A}_{ij} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}A_{kl}$$

Then such A_{ij} are called the components of a covariant tensor of rank two.

1.5 MIXED TENSOR OF RANK TWO

Let $A^i_j(i,j = 1,2,...,n)$ be n^2 functions of the coordinates $x^1, x^2, ..., x^n$ in X-coordinate system. If the quantities A^i_j are transformed into $\overline{A^i_j}$ in the Y-coordinate system having the coordinates as $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$,then according to the law of transformation

$$\overline{A^i_j} = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \overline{x}^j}A^k_l$$

Then A^i_j are called the components of the mixed tensor of rank two.

Theorem 1.1 *The Kronecker delta is a mixed tensor of rank two.*

Proof:

Let X and Y be two coordinate systems. And let the component of Kronecker delta in the X-coordinate system be δ^i_j and the component of Kronecker delta *in the Y-coordinate system be* $\overline{\delta^i_j}$ *, then according to the law of transformation, we can write ,*

$$\overline{\delta^i_j} = \frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial x^k}{\partial x^l}$$

$$\overline{\delta^i_j} = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \overline{x}^j}\delta^k_l$$

This shows that Kronecker δ^i_j *is a mixed tensor of rank two.*

1.6 HIGHER RANK TENSORS

Consider N^2 *functions* A^{ij} *which are defined in* $x^1, x^2, ..., x^N$ *coordinate system. If due to a change in coordinate system* $\overline{x}^1, \overline{x}^2, ..., \overline{x}^n$*, the quantities*$A^{ij}$ *transform according to the eqn*

$$\overline{A}^{ij} = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial \overline{x}^j}{\partial x^m}A^{km} \tag{1.13}$$

Then we call A^{ij} *the components of a contravariant tensor of second order.*
Similarly , if the N^2 *functions* A_{ij} *transform according to the law*

$$\overline{A}_{ij} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^m}{\partial \overline{x}^j}A_{km} \tag{1.14}$$

We call A_{ij} *the components of a covariant tensor of second order*
Again, if we have N^2 *functions* A^i_j *which transform according to the equation*

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial \overline{x}^m}{\partial \overline{x}^j}A^k_m \tag{1.15}$$

We call A^i_j the components of a mixed tensor of second order.

A set of N^{s+p} quantities

$$A^{k_1k_2...k_s}_{m_1m_2...m_p}$$

is said to be the components of a mixed tensor of (s+p)th order, if they transform according to the equation

$$\overline{A}^{k_1k_2...k_s}_{m_1m_2...m_p} = \frac{\partial \overline{x}^k_1}{\partial x^t_1}...\frac{\partial \overline{x}^k_s}{\partial x^t_s}\frac{\partial x^q_1}{\partial \overline{x}^m_1}...\frac{\partial x^q_p}{\partial \overline{x}^m_p}A^{t_1t_2...t_s}_{q_1q_2...q_p}$$

1.7 SYMMETRIC AND SKEW SYMMETRIC TENSORS

Definition 1.11 *Symmetric Tensors:*

A tensor is said to be symmetric with respect to two contravariant (or two covariant) indices if its components remain unchanged on an interchange of the two indices

Example:

1. *The tensor A^{ij} is symmetric if $A^{ij} = A^{ji}$*

2. *The tensor A^{ijk}_{lm} is symmetric if $A^{ijk}_{lm} = A^{jik}_{lm}$*

Theorem 1.2 *A symmetric tensor of rank two has only $\frac{1}{2}n(n+1)$ different components in n dimensional space.*

Proof:

Let A^{ij} be a symmetric tensor of rank two , then $A^{ij} = A^{ji}$.

The components of A^{ij} *are* $\begin{bmatrix} A^{11} & A^{12} & A^{13} & \cdots & A^{1n} \\ A^{21} & A^{22} & A^{23} & \cdots & A^{2n} \\ A^{31} & A^{32} & A^{33} & \cdots & A^{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \cdots & A^{nn} \end{bmatrix}$

i.e., A^{ij} *will have* n^2 *components . Out of these* n^2 *components , n components* $A^{11}, A^{22}, A^{33}, ..., A^{nn}$ *are different. Thus remaining components are* $(n^2 - n)$*, in which* $A^{12} = A^{21}, A^{23} = A^{32}$*etc due to symmetry.*

So, the remaining different components are$\frac{1}{2}(n^2 - n)$*Hence the total number of different components*

$$= n + \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n+1)$$

Definition 1.12 *Skew-symmetric Tensor:*

A tensor is said to be skew-symmetric with respect to two contravariant (or two covariant) indices if its components change sign on interchange of the two indices

Example:

1. *The tensor* A^{ij} *is skew-symmetric if* $A^{ij} = -A^{ji}$

2. *The tensor* A^{ijk}_{lm} *is skw-symmetric if* $A^{ijk}_{lm} = -A^{jik}_{lm}$

Theorem 1.3 *A skew-symmetric tensor of rank two has only* $\frac{1}{2}n(n-1)$ *different non zero components .*

Proof:

Let A^{ij} be a skew-symmetric tensor of rank two , then $A^{ij} = -A^{ji}$.

The components of A^{ij} are $\begin{bmatrix} 0 & A^{12} & A^{13} & \cdots & A^{1n} \\ A^{21} & 0 & A^{23} & \cdots & A^{2n} \\ A^{31} & A^{32} & 0 & \cdots & A^{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \cdots & 0 \end{bmatrix}$

$[since, A^{ii} = -A^{ii} \Rightarrow 2A^{ii} = 0 \Rightarrow A^{ii} = 0 \Rightarrow A^{11} = A^{22} = ... = A^{nn} = 0]$ *i.e., A^{ij} will have n^2 components . Out of these n^2 components , n components $A^{11}, A^{22}, A^{33}, ..., A^{nn}$ are zero. Thus remaining components are $(n^2 - n)$, in which $A^{12} = -A^{21}, A^{23} = -A^{32}$etc due to skew-symmetry.*

So, the remaining different components are$\frac{1}{2}(n^2 - n)$Hence the total number of different non-zero components

$$= \frac{1}{2}n(n-1)$$

Theorem 1.4 *A covariant or contravariant tensor of rank two, say, A_{ij}can always be written as a sum of a symmetric and a skew symmetric tensor.*

Proof:

Consider a covariant tensor A_{ij} . We can write A_{ij} as

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$$

$$A_{ij} = S_{ij} + T_{ij},$$

where

$$S_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$$

and

$$T_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$$

Now

$$, S_{ji} = \frac{1}{2}(A_{ji} + A_{ij}) \quad ; S_{ji} = S_{ij}$$

So S_{ij} *is symmetric tensor.*

and

$$T_{ij} = \frac{1}{2}(A_{ij} - A_{ji}$$

$$T_{ji} = \frac{1}{2}(A_{ji} - A_{ij})$$

$$= -\frac{1}{2}(A_{ij} - A_{ji})$$

$$T_{ji} = -T_{ij} \quad or$$

$$T_{ij} = -T_{ji}$$

So, T_{ij} *is skew symmetric tensor.*

Chapter 2

TENSOR ALGEBRA

2.1 ADDITION AND SUBTRACTION OF TENSORS

Two tensors are said to be of same type if they have same number of contravariant and covariant indices.Thus A^i_{jk} *and* B^i_{jk} *are same type of tensors.Such tensors can be added and subtracted to produce a single tensor.Hence the sum and diffrence of the tensors* A^i_{jk} *and* B^i_{jk} *are respectively the tensors*

$$C^i_{jk} = A^i_{jk} + B^i_{jk}$$

$$D^i_{jk} = A^i_{jk} - B^i_{jk}$$

Theorem 2.1 *If* A^{ij}_k *and* B^{lm}_n *are tensors then,their sum and difference are tensors of the same rank and type.*

proof:

Since A^i_{jk} *and* B^i_{jk} *are tensors , then,according to the law of transformation,*

$$\overline{A}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}A^{pq}_r$$

and

$$\overline{B}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}B^{pq}_r$$

Then,

$$\overline{A}^{ij}_k \pm \overline{B}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}(A^{pq}_r \pm \ B^{pq}_r)$$

If

$$\overline{A}^{ij}_{k} \pm \overline{B}^{ij}_{k} = \overline{C}^{ij}_{k}$$

$$(A^{pq}_{r} \pm \ B^{pq}_{r}) = C^{pq}_{r}$$

So,

$$\overline{C}^{ij}_{k} = \frac{\partial \overline{x}^{i}}{\partial x^{p}} \frac{\partial \overline{x}^{j}}{\partial x^{q}} \frac{\partial x^{r}}{\partial \overline{x}^{k}} C^{pq}_{r}$$

This shows that C^{ij}_{k} is a tensor of same rank and type as that of A^{ij}_{k} and B^{ij}_{k}

Theorem 2.2 *The sum(or difference) of two tensors which have the same number of covariant and the same contravariant indices is again a tensor of the same rank and type as the given tensors.*

Proof:

Consider the two tensors $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ and $B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ of the same rank and type.

Then by the law of transformation,

$$\overline{A}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} \ldots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} \ldots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$$

and ,

$$\overline{B}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} \ldots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} \ldots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$$

.Then ,

$$\overline{A}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \pm \overline{B}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} \ldots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} \ldots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} (A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} \pm \ B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s})$$

If.,

$$\overline{A}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \pm \overline{B}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \overline{C}^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$$

and

$$A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} \pm B^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$$

So.,

$$\overline{C}^{i_1 i_2 ... i_r}_{j_1 j_2 ... j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} ... \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} ... \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} C^{p_1 p_2 ... p_r}_{q_1 q_2 ... q_s}$$

It is the law of transformation of a mixed tensor of rank r+s

So, $\overline{C}^{i_1 i_2 ... i_r}_{j_1 j_2 ... j_s}$ is a mixed tensor of rank r+s or of type (r,s).

2.2 MULTIPLICATION OF TENSORS

Definition 2.1 *Outerproduct:The outer product of two tensors denoted by B^{ij}_{klm} and C^{qrs}_{uvwx} is defined as*

$$A^{ijqrs}_{klmuvwx} = B^{ij}_{klm} C^{qrs}_{uvwx}$$

NOTE:

The division of a tensor of rank greater than zero by another tensor of rank greater than zero is not defined.

Theorem 2.3 *The multiplication of two tensors is a tensor whose rank is the sum of the ranks of the two tensors.*

Proof:

Consider two tensors $A^{i_1 i_2 .... i_r}_{j_1 j_2 ... j_s}$, $B^{i_1 i_2 .... i_r}_{j_1 j_2 ... j_s}$

Then by the law of transformation,

$$\overline{A}^{i_1 i_2 ... i_r}_{j_1 j_2 ... j_s} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} ... \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} ... \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} A^{p_1 p_2 ... p_r}_{q_1 q_2 ... q_s}$$

and,

$$\overline{B}^{k_1 k_2 ... k_m}_{l_1 l_2 ... l_n} = \frac{\partial \overline{x}^{k_1}}{\partial x^{\alpha_1}} \frac{\partial \overline{x}^{k_2}}{\partial x^{\alpha_2}} ... \frac{\partial \overline{x}^{k_m}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial \overline{x}^{l_1}} \frac{\partial x^{\beta_2}}{\partial \overline{x}^{l_2}} ... \frac{\partial x^{\beta_n}}{\partial \overline{x}^{l_n}} B^{\alpha_1 \alpha_2 ... \alpha_m}_{\beta_1 \beta_2 ... \beta_n}$$

Then their product is,

$$\overline{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \overline{B}^{k_1 k_2 \dots k_m}_{l_1 l_2 \dots l_n} =$$

$$\frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} \cdots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} \frac{\partial \overline{x}^{k_1}}{\partial x^{\alpha_1}} \frac{\partial \overline{x}^{k_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial \overline{x}^{k_m}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial \overline{x}^{l_1}} \frac{\partial x^{\beta_2}}{\partial \overline{x}^{l_2}} \cdots \frac{\partial x^{\beta_n}}{\partial \overline{x}^{l_n}} A^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} B^{\alpha_1 \alpha_2 \dots \alpha_m}_{\beta_1 \beta_2 \dots \beta_n}$$

If

$$\overline{C}^{i_1 i_2 \dots i_r k_1 k_2 .. k_m}_{j_1 j_2 \dots j_s l_1 l_2 .. l_n} = \overline{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} \overline{B}^{k_1 k_2 \dots k_m}_{l_1 l_2 \dots l_n}$$

and,

$$C^{p_1 p_2 \dots p_r \alpha_1 \alpha_2 .. \alpha_m}_{q_1 q_2 \dots q_s \beta_1 \beta_2 .. \beta_n} = A^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} B^{\alpha_1 \alpha_2 \dots \alpha_m}_{\beta_1 \beta_2 \dots \beta_n}$$

So,

$$\overline{C}^{i_1 i_2 \dots i_r k_1 k_2 .. k_m}_{j_1 j_2 \dots j_s l_1 l_2 .. l_n} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{p_2}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{q_2}}{\partial \overline{x}^{j_2}} \cdots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} \frac{\partial \overline{x}^{k_1}}{\partial x^{\alpha_1}} \frac{\partial \overline{x}^{k_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial \overline{x}^{k_m}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial \overline{x}^{l_1}} \frac{\partial x^{\beta_2}}{\partial \overline{x}^{l_2}} \cdots \frac{\partial x^{\beta_n}}{\partial \overline{x}^{l_n}} C^{p_1 p_2 \dots p_r \alpha_1 \alpha_2 .. \alpha_m}_{q_1 q_2 \dots q_s \beta_1 \beta_2 .. \beta_n}$$

This is law of transformation of a mixed tensor of rank r + m + s + n. So, $\overline{C}^{i_1 i_2 \dots i_r k_1 k_2 .. k_m}_{j_1 j_2 \dots j_s l_1 l_2 .. l_n}$ *is a mixed tensor of rank r + m + s + n or of type (r + m, s + n). Such product is called outer product or open proudct of two tensors.*

Example 2.1 *:*

If A^i *and* B_j *are the components of a contravariant and covariant tensors of rank one, then prove that* $A^i B_j$ *are components of a mixed tensor of rank two.*

Solution:

As A^i *is contravariant tensor of rank one and* B_j *is covariant tensor of rank one.*

Then according to the law of transformation,

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^k} A^k \tag{2.1}$$

$$\overline{B}_j = \frac{\partial x^l}{\partial \overline{x}^j} B_l \tag{2.2}$$

Multiply (2.1) and (2.2), we get,

$$\overline{A}^i \overline{B}_j = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \overline{x}^j} A^k B_l$$

This is law of transformation of tensor of rank two. So, $\mathrm{A}^i B_j$ are mixed tensor of rank two.

Example 2.2 *:*

Show that the product of two tensors A^i_j and B^{kl}_m is a tensor of rank five.
Solution:

Since,A^i_j and B^{kl}_m are tensors,
by law of transformation,

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} A^p_q$$

and

$$\overline{B}^{kl}_m = \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} B^{rs}_t$$

Multipying these,we get,

$$\overline{A}^i_j \overline{B}^{kl}_m = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} A^p_q B^{rs}_t$$

This is law of transformation of tensor of rank five. So, $A^i_j B^{kl}_m$ is a tensor of rank five.

2.3 CONTRACTION OF A TENSOR

Definition 2.2 *The process of getting a tensor of lower order (reduced by 2) by putting a covariant index equal to a contravariant index and performing* *the summation indicated is known as Contraction.*

In other words, if in a tensor we put one contravariant and one covariant indices equal, the process is called contraction of a tensor.

Example 2.3 *: Consider a mixed tensor A^{ijk}_{lm} of order five by law of transformation,*

$$\overline{A}^{ijk}_{lm} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \overline{x}^l}\frac{\partial x^t}{\partial \overline{x}^m}A^{pqr}_{st}$$

Put the covariant index l = contravariant index i, so that,

$$\overline{A}^{ijk}_{lm} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial x^t}{\partial \overline{x}^m}A^{pqr}_{st}$$

$$= \frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial x^s}{\partial x^p}\frac{\partial x^t}{\partial \overline{x}^m}A^{pqr}_{st}$$

$$= \frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial x^t}{\partial \overline{x}^m}\delta^s_p A^{pqr}_{st}$$

(Since, $\frac{\partial x^s}{\partial x^p} = \delta^s_p$*)*

$$\overline{A}^{ijk}_{lm} = \frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial \overline{x}^k}{\partial x^r}\frac{\partial x^t}{\partial \overline{x}^m}A^{pqr}_{pt}$$

This is law of transformation of tensor of rank 3. So, A^{ijk}_{lm} is a tensor of rank 3 and type (1, 2) while A^{ijk}_{lm} is a tensor of rank 5 and type (2, 3).
i.e, that contraction reduces rank of tensor by two.

2.4 INNER PRODUCT OF TWO TENSORS

Definition 2.3 *Consider the tensors A^{ij}_k and B^l_{mn} . First form their outer product $A^{ij}_k B^l_{mn}$ and contract this by putting $l = k$ then the resultant is also a tensor, which is called the inner product of the given tensors.*
Hence the inner product of two tensors is obtained by first taking outer product and then contracting it.

Example 2.4 *: If A^i and B_i are the components of a contravariant and covariant tensors of rank are respectively then prove that A^iB_i is scalar or* *invariant.*

Solution :

Since, A^i *and* B_i *are the components of a contravariant and covariant tensors of rank one respectively,*

then by law of transformation,

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^p} A^p$$

and

$$\overline{B}_i = \frac{\partial x^q}{\partial \overline{x}^i} B_q$$

Multiplying these,we get,

$$\overline{A}^i \overline{B}_i = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^p} A^p B_q$$

$$= \frac{\partial x^q}{\partial x^p} A^p B_q$$

$$= \delta^q_p A^p B_q$$

(Since, $\frac{\partial x^q}{\partial x^p} = \delta^q_p$*)*

$$= A^p B_p$$

i.e,

$$\overline{A}^i \overline{B}_i = A^p B_p$$

This shows that $A^i B_i$ *is scalar or Invariant.*

Example 2.5 *: If* A^i_j*is mixed tensor of rank 2 and* B^{kl}_m *is mixed tensor of rank 3,Prove that is a mixed* $A^i_j B^{jl}_m$ *tensor of rank 3.*

Solution :

As A^i_j*is mixed tensor of rank 2 and* B^{kl}_m *is mixed tensor of rank 3,*

by law of transformation,

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} A^p_q$$

and

$$\overline{B}^{kl}_m = \frac{\partial \overline{x}^k}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} B^{rs}_t$$

put k=j,then,

$$\overline{B}^{jl}_m = \frac{\partial \overline{x}^j}{\partial x^r} \frac{\partial \overline{x}^l}{\partial x^s} \frac{\partial x^t}{\partial \overline{x}^m} B^{rs}_t$$

Then,

$$\overline{A}^i_j\overline{B}^{jl}_m = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial x^r}\frac{\partial \overline{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \overline{x}^m}A^p_q B^{rs}_t$$
$$= \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \overline{x}^m}\delta^q_r A^p_q B^{rs}_t$$

(Since, $\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial x^r} = \frac{\partial x^q}{\partial x^r} = \delta^q_r$*)*
i.e,

$$\overline{A}^i_j\overline{B}^{jl}_m = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^l}{\partial x^s}\frac{\partial x^t}{\partial \overline{x}^m}A^p_q B^{qs}_t$$

(Since, $\delta^q_r B^{rs}_t = B^{qs}_t$*)*

This is the law of transformation of a mixed tensor of rank three. Hence $A^i_j B^{jl}_m$ *is a mixed tensor of rank three.*

2.5 QUOTIENT LAW

Using this law, we can test whether a given quantity is a tensor or not. Suppose that a quantity A is given and we have to check whether A is a tensor or not.
For this, we take the inner product of A with an arbitrary tensor, if this inner product is a tensor then A is also a tensor.

STATEMENT:

If the inner product of a set of functions with an arbitrary tensor, is a tensor, then these set of functions are the components of a tensor.

Example 2.6 *:*

Show that the expression $A(i,j,k)$ *is a covariant tensor of rank three if* $A(i,j,k)B^k$ *is covariant tensor of rank two and* B^k *is contravariant vector*

Solution:

Let X and Y be two coordinate systems.
As given $A(i,j,k)B^k$ is covariant tensor of rank two then,

$$\overline{A}(i,j,k)\overline{B}^k = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A(p,q,r)B^r \tag{2.3}$$

Since

B^k is a contravariant vector,then,

$$\overline{B}^k = \frac{\partial \overline{x}^k}{\partial x^r}B^r$$

or

$$B^r = \frac{\partial x^r}{\partial \overline{x}^k}\overline{B}^k$$

From (2.3),

$$\overline{A}(i,j,k)\overline{B}^k = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A(p,q,r)\frac{\partial x^r}{\partial \overline{x}^k}\overline{B}^k$$

$$\overline{A}(i,j,k)\overline{B}^k = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k}A(p,q,r)\overline{B}^k$$

$$A(i,j,k) = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k}A(p,q,r)$$

i.e,$A(i,j,k)$ is covariant tensor of rank three.

Example 2.7 *:*

If $A(i,j,k)A^iB^jC_k$ is a scalar for arbitrary vectors A_i, B^j, C_k. Show that $A(i,j,k)$ is a tensor of type (1, 2).
Solution:

Let X and Y be two coordinate systems. As given is scalar. Then $A(i,j,k)A^iB^jC_k$ is a scalar,

$$\overline{A}(i,j,k)\overline{A}^i\overline{B}^j\overline{C}_k = A(p,q,r)A^pB^qC_r \tag{2.4}$$

Since

A^i, B^j and C_k are vectors ,then,

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^p}A^p \qquad or \qquad A^p = \frac{\partial x^p}{\partial \overline{x}^i}\overline{A}^i$$

$$\overline{B}^j = \frac{\partial \overline{x}^j}{\partial x^q} B^q \qquad or \qquad or B^q = \frac{\partial x^q}{\partial \overline{x}^j}\overline{B}^j$$

$$\overline{C}^k = \frac{\partial \overline{x}^k}{\partial x^r} C^r \qquad or \qquad C^r == \frac{\partial x^q}{\partial \overline{x}^j}\overline{B}^j$$

Substituting in (2.4)

$$\overline{A}(i,j,k)\overline{A}^i\overline{B}^j\overline{C}_k = A(p,q,r)\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k}\overline{B}^j\overline{A}^i\overline{B}^j\overline{C}_k$$

$$\overline{A}(i,j,k) = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k}A(p,q,r)$$

i.e,$A(i,j,k)$ is tensor of type (1, 2).

2.6 CONJUGATE (RECIPROCAL) SYMMETRIC TENSOR

Consider a covariant symmetric tensor A_{ij} of rank two. Let 'd' denote the determinant $|A_{ij}|$with the elements A_{ij}.
$i.e, d = |A_{ij}|$ and $d \neq 0$.
Now,define A^{ij} by,

$$A^{ij} = \frac{Cofactor of A_{ij}}{d}$$

A^{ij} is a contravariant symmetric tensor of rank two which is called conjugate (or Reciprocal) tensor of A_{ij}.

Theorem 2.4 *If B_{ij} is the cofactor of A_{ij} in the determinant $d = |A_{ij}| \neq 0$and A_{ij} defined as,*

$$A^{ij} = \frac{B_{ij}}{d}$$

$$Then, A_{ij}A^{kj} = \delta_i^k$$

Proof:
From the properties of the determinants, we have two results

1. $A_{ij}B_{ij} = d$

$\Rightarrow A_{ij}\frac{B_{ij}}{d} = 1$

$\Rightarrow A_{ij}A^{ij} = 1$ $\qquad (Since, A^{ij} = \frac{d}{B_{ij}})$

2. $A_{ij}B_{kj} = 0$

$\Rightarrow A_{ij}\frac{B_{kj}}{d} = 0$ $\qquad$ (since $d \neq 0$)

$\Rightarrow A_{ij}A^{kj} = 0$ $\qquad$ (if $i \neq k$)

From 1 and 2,

$$A_{ij}A^{kj} = \begin{cases} 1 & if \quad i = k \\ 0 & if \quad i \neq k \end{cases}$$

i.e, $A_{ij}A^{kj} = \delta_i^k$

Chapter 3

METRIC TENSOR & REIMANNIAN METRIC

3.1 THE METRIC TENSOR

Definition 3.1 *In rectangular cartesian coordinates, the distance between two neighbouring point are* (x, y, z) *and* $(x + dx, y + dy, z + dz)$*is given by* $ds^2 = dx^2 + dy^2 + dz^2$ *.*

In n-dimensional space, Riemann defined the distance ds between two neighbouring points x^i *and* $x^i + dx^i (i = 1, 2, ...n)$ *by quadratic differential form,*

$$ds^2 = g_{11}(dx^1)^2 + g_{12}dx^1dx^2 + \cdots + g_{1n}dx^1dx^n$$
$$+g_{21}dx^2dx^1 + g_{22}(dx^2)^2 + \cdots + g_{2n}dx^2dx^n$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+g_{n1}dx^ndx^1 + g_{n2}dx^ndx^2 + \cdots + g_{nn}(dx^n)^2$$

i.e,

$$ds^2 = g_{ij}dx^idx^j (i, j = 1, 2, ..., n) \tag{3.1}$$

,using summation convention.

Where g_{ij} *are the functions of the coordinates* x^i*such that*

$$g = |g_{ij}| \neq 0$$

The quadratic differential form (3.1) is called the Riemannian Metric or Metric or line element for n-dimensional space and such n-dimensional space is called Riemannian space and denoted by V_n and g_{ij} is called Metric Tensor or Fundamental tensor.
The geometry based on Riemannian Metric is called the Riemannian Geometry.

Theorem 3.1 *The Metric tensor g_{ij} is a covariant symmetric tensor of rank two.*

Proof :

The metric is given by,

$$ds^2 = g_{ij}dx^i dx^j (i, j = 1, 2, ..., n) \tag{3.2}$$

Let x^i be the coordinates in X-coordinate system and $\overline{x}^i$ be the coordinates in Y-coordinate system. Then metric $ds^2 = g_{ij}dx^i dx^j$ transforms to $ds^2 = \overline{g}_{ij}d\overline{x}^i d\overline{x}^j$
Since distance being scalar quantity,

$$ds^2 = g_{ij}dx^i dx^j = \overline{g}_{ij}d\overline{x}^i d\overline{x}^j \tag{3.3}$$

The theorem will be proved in three steps.

1. To show dx^j is a contravariant vector
   If ,$\overline{x}^i = \overline{x}^i(x^1 x^2, ...x^n)$,

$$\begin{aligned} \mathrm{d}\overline{x}^i &= \frac{\partial \overline{x}^i}{\partial x^1}dx^1 + \frac{\partial \overline{x}^i}{\partial x^2}dx^2 + ... + \frac{\partial \overline{x}^i}{\partial x^n}dx^n \\ d\overline{x}^i &= \frac{\partial \overline{x}^i}{\partial x^k}dx^k \end{aligned} \tag{3.5}$$

   It is law of transformation of contravariant vector. So, dx^i is contravariant vector.

2. To show that g_{ij} is a covariant tensor of rank two.

   Since,

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^k} dx^k \qquad and \qquad d\overline{x}^j = \frac{\partial \overline{x}^j}{\partial x^l} dx^l$$

   from equation(3.3)

$$g_{ij} dx^i dx^j = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^k} dx^k \frac{\partial \overline{x}^j}{\partial x^l} dx^l$$

$$g_{ij} dx^i dx^j = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^l} dx^k dx^l$$

$$g_{kl} dx^k dx^l = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^l} dx^k dx^l$$

   Since, $g_{ij} dx^i dx^j = g_{kl} dx^k dx^l$ (i,j are dummy indices)

$$\left[g_{kl} - \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^l} \right] dx^k dx^l = 0$$

$$g_{kl} - \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^l} = 0$$

   as dx^k and dx^l are arbitrary.

$$g_{kl} = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^l}$$

or,

$$\overline{g}_{ij} = g_{kl} \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j}$$

So, g_{ij} is covariant tensor of rank two.

3. To show that g_{ij} is symmetric. Then g_{ij} can be written as,

$$g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) + \frac{1}{2}(g_{ij} - g_{ji})$$

$$g_{ij} = A_{ij} + B_{ij}$$

where

$$A_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) = symmetric$$

$$B_{ij} = \frac{1}{2}(g_{ij} - g_{ji}) = skew - symmetric$$

Now

$$g_{ij} dx^i dx^j = (A_{ij} + B_{ij}) dx^i dx^j$$

$$(g_{ij} - A_{ij})dx^i dx^j = B_{ij}dx^i dx^j \tag{3.6}$$

Interchanging the dummy indices in $B_{ij}dx^i dx^j$,we have,

$$B_{ij}dx^i dx^j = B_{ji}dx^i dx^j$$

$$B_{ij}dx^i dx^j = -B_{ij}dx^i dx^j$$

Since B_{ij} is skew-symmetric,i.e, B_{ij}=-B_{ij}

$$B_{ij}dx^i dx^j + B_{ij}dx^i dx^j = 0$$

$$2B_{ij}dx^i dx^j = 0$$

$$B_{ij}dx^i dx^j = 0$$

So from(3.6),

$$(g_{ij} - A_{ij})dx^i dx^j = 0$$

$$\Rightarrow g_{ij} = A_{ij}$$

as dx^i, dx^j are arbitrary.

So, $g_{ij}$is symmetric since A_{ij} is symmetric. Hence g_{ij} is a covariant symmetric tensor of rank two. This is called fundamental Covariant Tensor.

Example 3.1 *: Show that $g_{ij}dx^i dx^j$ is an invariant.*

Proof:

Let x^i be coordinates of a point in X-coordinate system and $\overline{x}^i$ be coordinates of a same point in Y-coordinate system.

Since g_{ij} is a Covariant tensor of rank two.

Then,

$$\overline{g}_{ij} = g_{kl}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}$$

$$\overline{g}_{ij} - g_{kl}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j} = 0$$

$$\left[\overline{g}_{ij} - g_{kl}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\right] dx^i dx^j = 0$$

$$\overline{g}_{ij}dx^i dx^j = g_{kl}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}dx^i dx^j$$
$$\overline{g}_{ij}dx^i dx^j = g_{kl}\frac{\partial x^k}{\partial \overline{x}^i}dx^i\frac{\partial x^l}{\partial \overline{x}^j}dx^j$$

i.e,

$$\overline{g}_{ij}dx^i dx^j = g_{kl}dx^k dx^l$$

So, $g_{ij}dx^i dx^j$ *is an ivariant.*

3.2 CONJUGATE METRIC TENSOR

Definition 3.2 *The conjugate Metric Tensor to* g_{ij} *, which is written as* g^{ij} *, is defined by*

$$g^{ij} = \frac{B_{ij}}{g}$$

whereB_{ij} *is the cofactor of* g_{ij} *in the determinant* $= g = |g_{ij}| \neq 0$
By theorem (2.4),

$$A_{ij}A^{kj} = \delta_i^k$$

so,

$$g_{ij}g^{kj} = \delta_i^k$$

NOTE:
(i) Tensors g_{ij} and g^{ij} are Metric Tensor or Fundamental Tensors.
(ii) g_{ij} is called first fundamental Tensor and g^{ij} second fundamental Tensors

Example 3.2 *:*

Find the Metric and component of first and second fundamental tensor is cylindrical coordinates.

Solution:

Let (x^1, x^2, x^3) *be the Cartesian coordinates and* $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$*be the cylindrical coordinates of a point. The cylindrical coordinates are given by,*

$$x = r\cos\theta, y = r\sin\theta, z = z$$

So that,

$$x^1 = x, x^2 = y, x^3 = z \qquad and \qquad \overline{x}^1 = r, \overline{x}^2 = \theta, \overline{x}^3 = z \tag{3.7}$$

Let g_{ij} and $\overline{g}_{ij}$ be the metric tensors in Cartesian coordinates and cylindrical coordinates respectively.

The metric in Cartesian coordinate is given by,

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{3.8}$$

but,

$$ds^2 = g_{ij}dx^i dx^j (i, j = 1, 2, 3)$$

$$ds^2 = g_{11}(dx^1)^2 + g_{12}dx^1dx^2 + g_{13}dx^1dx^3$$
$$+g_{21}dx^2dx^1 + g_{22}(dx^2)^2 + +g_{23}dx^2dx^3$$
$$+g_{31}dx^3dx^1 + g_{32}dx^3dx^2 + +g_{33}(dx^3)^3 \tag{3.9}$$

Comparing (3.8) and (3.9), we have,

$g_{11} = g_{22} = g_{33} = 1$ and $g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

On transformation,$\overline{g}_{ij} = g_{ij}\frac{\partial x^i}{\partial \overline{x}^i}\frac{\partial x^j}{\partial \overline{x}^j}$, since,$g_{ij}$ is a covariant tensor of rank two(i,j=1,2,3)

$$\overline{g}_{ij} = g_{11}\frac{\partial x^1}{\partial \overline{x}^i}\frac{\partial x^1}{\partial \overline{x}^j} + g_{22}\frac{\partial x^2}{\partial \overline{x}^i}\frac{\partial x^2}{\partial \overline{x}^j} + g_{33}\frac{\partial x^3}{\partial \overline{x}^i}\frac{\partial x^3}{\partial \overline{x}^j}$$

Since,i,j are dummy indices,

Put i=j=1

Since,$g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x^1}{\partial \overline{x}^1}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \overline{x}^1}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \overline{x}^1}\right)^2$$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x}{\partial r}\right)^2 + g_{22}\left(\frac{\partial y}{\partial r}\right)^2 + g_{33}\left(\frac{\partial z}{\partial r}\right)^2$$

Since,$x = r\cos\theta, y = r\sin\theta, z = z$

$$\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial z}{\partial r} = 0$$

and $g_{11} = g_{22} = g_{33} = 1$

$$\Rightarrow \overline{g}_{11} = \cos^2\theta + \sin^2\theta + 0$$

$$\overline{g}_{11} = 1$$

Put i=j=2

Since,$g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

$$\overline{g}_{22} = g_{11}\left(\frac{\partial x^1}{\partial \overline{x}^2}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \overline{x}^2}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \overline{x}^2}\right)^2$$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x}{\partial \theta}\right)^2 + g_{22}\left(\frac{\partial y}{\partial \theta}\right)^2 + g_{33}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Since,$x = r\cos\theta, y = r\sin\theta, z = z$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta, \frac{\partial z}{\partial \theta} = 0$$

and $g_{11} = g_{22} = g_{33} = 1$

$$\Rightarrow \overline{g}_{22} = (-r\sin\theta)^2 + (r\cos\theta)^2 + 0$$

$$\overline{g}_{22} = r^r\sin^2\theta + r^r\cos^2\theta$$

$$\overline{g}_{22} = r^r$$

Put i=j=3

Since,$g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

$$\overline{g}_{33} = g_{11}\left(\frac{\partial x^1}{\partial \overline{x}^3}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \overline{x}^3}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \overline{x}^3}\right)^2$$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x}{\partial z}\right)^2 + g_{22}\left(\frac{\partial y}{\partial z}\right)^2 + g_{33}\left(\frac{\partial z}{\partial z}\right)^2$$

Since,$x = r\cos\theta, y = r\sin\theta, z = z$

$$\frac{\partial x}{\partial z} = 0, \frac{\partial y}{\partial z} = 0, \frac{\partial z}{\partial z} = 1$$

and $g_{11} = g_{22} = g_{33} = 1$

$$\Rightarrow \overline{g}_{33} = 1$$

So,$\overline{g}_{11} = 1, \overline{g}_{22} = r^2, \overline{g}_{33} = 1$

and

$\overline{g}_{12} = \overline{g}_{13} = \overline{g}_{21} = \overline{g}_{23} = \overline{g}_{31} = \overline{g}_{32} = 0$

(i) The metric in cylindrical coordinates,

$$ds^2 = \overline{g}_{ij} d\overline{x}^i d\overline{x}^j (i, j = 1, 2, 3)$$

Since,

$$\overline{g}_{12} = \overline{g}_{13} = \overline{g}_{21} = \overline{g}_{23} = \overline{g}_{31} = \overline{g}_{32} = 0$$

$$\Rightarrow ds^2 = \overline{g}_{11}(d\overline{x}^1)^2 + \overline{g}_{22}(d\overline{x}^2)^2 + \overline{g}_{33}(d\overline{x}^3)^2$$

$$ds^2 = dr^2 + r^r(d\theta)^2 + dz^2$$

(ii) The first fundamental tensor is

$$\overline{g}_{ij} = \begin{bmatrix} \overline{g}_{11} & \overline{g}_{12} & \overline{g}_{13} \\ \overline{g}_{21} & \overline{g}_{22} & \overline{g}_{23} \\ \overline{g}_{31} & \overline{g}_{32} & \overline{g}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since,

$$g = |\overline{g}_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow g = r^2$$

(iii) The cofactor of g are given by,

$B_{11} = r^2, B_{22} = 1, B_{33} = r^2$ and, $B_{12} = B_{21} = B_{13} = B_{23} = B_{32} = 0$
The second fundamental tensor or conjugate tensor is,$g^{ij} = \frac{B^{ij}}{g}$

$$g^{11} = \frac{cofactor \quad of \quad g_{11} \quad in \quad g}{g}$$

$$g^{11} = \frac{B_{11}}{g} = \frac{r^2}{r^2} = 1$$

$$g^{22} = \frac{B_{22}}{g} = \frac{1}{r^2}$$

$$g^{33} = \frac{B_{33}}{g} = \frac{r^2}{r^2} = 1$$

and,

$$g^{12} = g^{13} = g^{21} = g^{23} = g^{31} = g^{32} = 0$$

Hence the second fundamental tensor in matrix form is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3.3 : *Find the matrix and component of first and second fundamental tensors in spherical coordinates.*

Solution:

Let (x^1, x^2, x^3) *be the cartesian coordinates and* $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ *be the spherical coordinates of a point. The spherical coordinates are given by*

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

So that,

$$x^1 = x, \quad x^2 = y, \quad x^3 = z \quad and \quad \overline{x}^1 = r, \quad \overline{x}^2 = \theta, \quad \overline{x}^3 = \phi$$

Let g_{ij} and $\overline{g}_{ij}$ be the metric tensors in cartesian and spherical coordinates respectively.
The metric in cartesian coordinates is given by,

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{3.10}$$

but,

$$ds^2 = g_{ij}dx^i dx^j (i, j = 1, 2, 3)$$

$$\begin{aligned} ds^2 = g_{11}(dx^1)^2 + g_{12}dx^1dx^2 + g_{13}dx^1dx^3 \\ +g_{21}dx^2dx^1 + g_{22}(dx^2)^2 + +g_{23}dx^2dx^3 \\ +g_{31}dx^3dx^1 + g_{32}dx^3dx^2 + +g_{33}(dx^3)^3 \end{aligned} \tag{3.11}$$

Comparing (3.10) and (3.11), we have,

$g_{11} = g_{22} = g_{33} = 1$ and $g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

On transformation,$\overline{g}_{ij} = g_{ij}\frac{\partial x^i}{\partial \overline{x}^i}\frac{\partial x^j}{\partial \overline{x}^j}$, since,$g_{ij}$ is a covariant tensor of rank two(i,j=1,2,3)

$$\overline{g}_{ij} = g_{11}\frac{\partial x^1}{\partial \overline{x}^i}\frac{\partial x^1}{\partial \overline{x}^j} + g_{22}\frac{\partial x^2}{\partial \overline{x}^i}\frac{\partial x^2}{\partial \overline{x}^j} + g_{33}\frac{\partial x^3}{\partial \overline{x}^i}\frac{\partial x^3}{\partial \overline{x}^j}$$

Since,i,j are dummy indices,

Put i=j=1

Since,$g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x^1}{\partial \overline{x}^1}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \overline{x}^1}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \overline{x}^1}\right)^2$$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x}{\partial r}\right)^2 + g_{22}\left(\frac{\partial y}{\partial r}\right)^2 + g_{33}\left(\frac{\partial z}{\partial r}\right)^2$$

Since,$x = r\sin\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial z}{\partial r} = \cos\theta$$

and $g_{11} = g_{22} = g_{33} = 1$

$$\Rightarrow \overline{g}_{11} = (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta$$

$$\overline{g}_{11} = 1$$

Put i=j=2

Since,$g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

$$\overline{g}_{22} = g_{11}\left(\frac{\partial x^1}{\partial \overline{x}^2}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \overline{x}^2}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \overline{x}^2}\right)^2$$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x}{\partial \theta}\right)^2 + g_{22}\left(\frac{\partial y}{\partial \theta}\right)^2 + g_{33}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Since,$x = r\sin\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

and $g_{11} = g_{22} = g_{33} = 1$

$$\Rightarrow \overline{g}_{22} = (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2$$

$$\overline{g}_{22} = r^r$$

Put i=j=3

Since,$g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = 0$

$$\overline{g}_{33} = g_{11}\left(\frac{\partial x^1}{\partial \overline{x}^3}\right)^2 + g_{22}\left(\frac{\partial x^2}{\partial \overline{x}^3}\right)^2 + g_{33}\left(\frac{\partial x^3}{\partial \overline{x}^3}\right)^2$$

$$\overline{g}_{11} = g_{11}\left(\frac{\partial x}{\partial \phi}\right)^2 + g_{22}\left(\frac{\partial y}{\partial \phi}\right)^2 + g_{33}\left(\frac{\partial z}{\partial \phi}\right)^2$$

Since,$x = r\sin\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi, \quad ,\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi, \quad \frac{\partial z}{\partial \phi} = 0$$

and $g_{11} = g_{22} = g_{33} = 1$

$$\Rightarrow \overline{g}_{33} = (-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0$$

$$\overline{g}_{33} = r^2\sin^2\theta$$

So,$\overline{g}_{11} = 1, \quad \overline{g}_{22} = r^2, \quad \overline{g}_{33} = r^2\sin^2\theta$

and

$\overline{g}_{12} = \overline{g}_{13} = \overline{g}_{21} = \overline{g}_{23} = \overline{g}_{31} = \overline{g}_{32} = 0$

(i) The metric in spherical coordinates,

$$ds^2 = \overline{g}_{ij}d\overline{x}^i d\overline{x}^j (i, j = 1, 2, 3)$$

Since,

$$\overline{g}_{12} = \overline{g}_{13} = \overline{g}_{21} = \overline{g}_{23} = \overline{g}_{31} = \overline{g}_{32} = 0$$

$$\Rightarrow ds^2 = \overline{g}_{11}(d\overline{x}^1)^2 + \overline{g}_{22}(d\overline{x}^2)^2 + \overline{g}_{33}(d\overline{x}^3)^2$$

$$ds^2 = dr^2 + r^r(d\theta)^2 + r^2\sin^2\theta d\phi^2$$

(ii) The first fundamental tensor is

$$\overline{g}_{ij} = \begin{bmatrix} \overline{g}_{11} & \overline{g}_{12} & \overline{g}_{13} \\ \overline{g}_{21} & \overline{g}_{22} & \overline{g}_{23} \\ \overline{g}_{31} & \overline{g}_{32} & \overline{g}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

Since,

$$g = |\overline{g}_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix}$$

$$\Rightarrow g = r^4\sin^2\theta$$

(iii) The cofactor of g are given by,

$B_{11} = 1, B_{22} = r^r, B_{33} = r^2 \sin^2\theta$ and, $B_{12} = B_{21} = B_{13} = B_{23} = B_{32} = 0$

The second fundamental tensor or conjugate tensor is,$g^{ij} = \frac{B^{ij}}{g}$

$$g^{11} = \frac{cofactor \quad of \quad g_{11} \quad in \quad g}{g}$$

$$g^{11} = \frac{B_{11}}{g} = \frac{r^4 \sin^2\theta}{r^4 \sin^2\theta} = 1$$

$$g^{22} = \frac{B_{22}}{g} = \frac{r^2 \sin^2\theta}{r^4 \sin^2\theta} = \frac{1}{r^2}$$

$$g^{33} = \frac{B_{33}}{g} = \frac{r^2}{r^4 \sin^2\theta} = \frac{1}{r^2 \sin^2\theta}$$

and,

$$g^{12} = g^{13} = g^{21} = g^{23} = g^{31} = g^{32} = 0$$

Hence the second fundamental tensor in matrix form is,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}$$

3.3 ASSOCIATED TENSOR

Definition 3.3 *A tensor obtained by the process of inner product of any tensor* $A^{i_1 i_2, \ldots i_r}_{j_1, j_2, \ldots, j_s}$ *with either of the fundamental tensor* g_{ij} *or* g^{ij} *is called associated tensor of given tensor.*

E.g : Consider a tensor A_{ijk} and form the following inner product

$$g^{\alpha i} A_{ijk} = A^{\alpha}_{jk}; \qquad g^{\alpha j} A_{ijk} = A^{\alpha}_{ik}; \qquad g^{\alpha k} A_{ijk} = A^{\alpha}_{ij}$$

All these tensors are called Associated tensor of A_{ijk}

ASSOCIATED VECTOR

Consider a covariant vector A_i . Then $g^{ik}A_i = A^k$ is called associated vector of A_i . Consider a contravariant vector A^j . Then $g_{jk}A^j = A_k$ is called associated vector of A^j

3.4 MAGNITUDE OF A VECTOR

The magnitude or length of contravariant vector A^i defined by,

$$A = \sqrt{g_{ij}A_iA_j}$$

or

$$A^2 = g_{ij}A_iA_j$$

A vector of magnitude one is called Unit vector. A vector of magnitude zero is called zero vector or Null vector

3.5 SCALAR PRODUCT OF TWO VECTORS

Let $\vec{A}$ and $\vec{B}$ be two vectors. Their scalar product is written as $\vec{A}.\vec{B}$ and defined by,

$$\vec{A}.\vec{B} = A^iB_i$$

Also,

$$\vec{A}.\vec{B} = A^iB_i = g_{ij}A^iB^j \quad since \quad B_i = g_{ij}B^j$$

$$\vec{A}.\vec{B} = A_iB^i = g^{ij}A_iB_j \quad since \quad B^i = g^{ij}B_j$$

Thus,

$$\vec{A}.\vec{A} = A^iA_i = g_{ij}A^iA^j = A^2$$

i.e,

$$A = |\vec{A}| = \sqrt{g_{ij}A^iA^j}$$

3.6 ANGLE BETWEEN TWO VECTORS

Let $\overline{A}$ and $\overline{B}$ be two vectors.Then,

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}$$

(Since, $|\vec{A}| = \sqrt{g_{ij}A^iA^j}$ and $|\vec{B}| = \sqrt{g_{ij}B^iB^j}$)

This is the required formula for $\cos\theta$,

Chapter 4

APPLICATIONS

1. Diffusion Tensor in MRI

   Diffusion MRI relies on the mathematics and physical interpretations of the geometric quantities known as tensors. Only a special case of the general mathematical notion is relevant to imaging, which is based on the concept of a symmetric matrix.

   Diffusion itself is tensorial, but in many cases the objective is not really about trying to study brain diffusion , but rather just trying to take advantage of diffusion anisotropy in white matter for the purpose of finding the orientation of the axons and the magnitude or degree of anisotropy. Tensors have a real, physical existence in a material or tissue so that they don't move when the coordinate system used to describe them is rotated. There are numerous different possible representations of a tensor (of rank 2), but among these, this discussion focuses on the ellipsoid because of its physical relevance to diffusion and because of its historical significance in the development of diffusion anisotropy imaging in MRI.

2. Multilinear subspace learning

   Multilinear subspace learning is an approach to dimensionality reduction.Dimensionality reduction can be performed on a data tensor whose observations have been vectorized and organized into a data tensor, or whose observations are matrices that are concatenated into a data tensor. Here are some examples of data tensors whose observations are vectorized or whose observations are matrices concatenated into data tensor images (2D/3D), video sequences (3D/4D), and hyperspectral cubes (3D/4D). The mapping from a high-dimensional vector space to a set of lower dimensional vector spaces is a multilinear projection.

Hyperspectral imaging, like other spectral imaging, collects and processes information from across the electromagnetic spectrum. The goal of hyperspectral imaging is to obtain the spectrum for each pixel in the image of a scene, with the purpose of finding objects, identifying materials, or detecting processes.

3. Electromagnetic tensor
   In electromagnetism, the electromagnetic tensor or electromagnetic field tensor (sometimes called the field strength tensor, Faraday tensor or Maxwell bivector) is a mathematical object that describes the electromagnetic field in spacetime. The field tensor was first used after the four-dimensional tensor formulation of special relativity was introduced by Hermann Minkowski.

4. Finite Deformation Tensor
   Deformation in continuum mechanics is the transformation of a body from a reference configuration to a current configuration. A configuration is a set containing the positions of all particles of the body.
   A deformation may be caused by external loads, body forces (such as gravity or electromagnetic forces), or changes in temperature, moisture content, or chemical reactions, etc.
   Strain is a description of deformation in terms of relative displacement of particles in the body that excludes rigid-body motions. Different equivalent choices may be made for the expression of a strain field depending on whether it is defined with respect to the initial or the final configuration of the body and on whether the metric tensor or its dual is considered.

CONCLUSION

Tensor analysis is a branch of mathematics concerned with relations or laws that are valid regardless of the system of co-ordinates used to specify the co-ordinates.

In this project we have studied about the different types of tensors and some basic concepts in Tensor analysis including tensor algebra and various properties regarding the metric tensor.The purpose of this project is to provide a brief knowledge in tensor analysis.Here, we consider mostly problems of tensors of rank two.Further extension of this project can be done by including more problems of higher rank tensors.

Tensors are important in physics as they provide a concise mathematical framework for formulating and solving physics problems in areas such as stress,elasticity,fluid mechanics and general relativity.

Tensors are a powerful mathematical tool that is used in many areas in engineering and physics including quantum mechanics,statistical thermodynamics,classical mechanics,electrodynamics,solid mechanics and fluid dynamics.

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