SPECTRA OF GRAPHS

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By

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CERTIFICATE

This is to certify that the dissertation entitled " SPECTRA OF GRAPHS " is a bonafide record of the work done by REENU ELIZABETH P.Kunder my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Smt.DHANALAKSHMI 0.M , Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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CONTENTS

INTRODUCTION

Technique from graph theory and linear algebra assist in studying the structure and enumeration of graphs. Eigen values of graphs mostly related to spectral graph theory starts by associating matrices to graphs notably adjacency and laplacian matrices.
The general theme is firstly to compute and estimate the eigen values of such matrices and secondly to relate the eigen values to structural properties of graphs.

This section opens the spectral perspectives on graphs.The spectrum is the list of distinct eigenvalues with their multiplicities.
The most important matrices associated to a graph are the adjacency matrix and laplacian matrix.Both are square matrix indexed by the vertex set V.

The adjacency matrix, A, is an n×n matrix where n = $\mid G \mid$ that represents which vertices are connected by an edge. If vertex i and vertex j are adjacent then $a_{ij} = 1$. If vertex i and vertex j are not adjacent then $a_{ij} = 0$. If G is a simple graph then $a_{ii} = 0$ for ∀ i because there are no loops. Also, because simple implies undirected, $a_{ij} = a_{ji}$ ∀ i,j ∈ V .

Two matrices are related by the formula $A + L = diag(deg)$,where $dia(deg)$denotes the diagonal matrix recording the degrees.We often view these matrices as operators on l^2V i.e., a finite dimensional space of complex valued functions on V ,endowed with the inner product
$< f, g >=\sum_v$ f(u) $\overline{g(v)}$.
Both are real symmetric.

We will make use of the tools throughout the following chapters and deal with the eigen values of graph and presents basic properties associated with the two type of matrices.Also we start with basic definitions and results from the theory of graph spectra.

In chapter 2, we determine the spectrum of a circulant matrix,also we compute the spectra of some special graphs . All graphs in this section are finite,undirected and simple.
Chapter 3, deals with the recent applications of eigenvalues and spectra of graphs,in the field of chemistry,applied science,graph coloring etc.Also,we determine one of the application of strongly regular graphs i.e.,the famous Friendship Theorem.

PRILIMINARIES

Graphs :

GRAPH :
It is an ordered tripple (V(G),E(G),ψ(G)) consisting of a non emplty set V(G) of vertices,a set E(G) of edges disjoint from V(G) and an incidence function ψ(G) and an incidence function ψ(G) that associates with each edge of G and unordered pair of vertices of G .

SIMPLE GRAPH :
A graph with no loops and parallel edges are called simple graph.

ISOMORPHISM OF GRAPH :
Let G and H are 2 graphs.The graph isomorphism from G to H is written as G≅H is a pair (ϕ,θ) where ϕ is a function from V(G) to V(H) and θ from E(G) to E(H) are bijections with property ψ_G (e) ={u,v} $\Rightarrow$ $\psi_H(\theta(e))$ ={$\phi(u)$,$\phi(v)$} .

COMPLETE GRAPH :
A simple graph G is said to complete if its each pair of distinct vartices is joined by an edge.A complete graph with n vertices is denoted by K_n and has exactly $\frac{n(n-1)}{2}$ edges.

BI-PARTITE GRAPH :
A graph is bi-paratite if its vertex set can be partitioned into two non empty subsets X and Y where each edge has one end in X and other end is Y.Such a partition (X,Y) is called a bi-partition of the graph.

COMPLETE BI-PARTITE GRAPH:
It is a simple bi-paratite graph with bi-partition(X,Y) in which each vertex of X is joined to each vertex of Y.

SUBGRAPH :
A graph H is a subgraph of G (H⊆G)if V(H)⊆V(G) and E(H)⊆E(G)and is a restriction of $\psi(G)$ to E(H).

REGULAR GRAPH:
A graph is called k-regular if every vertex of G has degree k.A graph is said to be regular if it is k-regular for some integer k>0.

CONNECTEDNESS :
Two vertices U and V of G are said to be connected if there is a U-V path in G.and a graph G is connected if every 2 vertices are connected i.e,a path joining every 2 vertices of G.

CYCLE :
A cycle is a closed trial in which all the vertices are distinct.

1.Let G be a graph U∈V. The number of edges incident at v in G is called the degree of the vertex and is denoted by d(v).The minimum and maximum of the degrees of the vertices of G is denoted by δ and Δ respectively

2. A walk in G is a alternating sequence W of vertices and edges starting and ending wih vertices in which v_{i-1} and v_i are the ends of e_i; it is a v_0-v_p walk.The integer P is the length of W.

3.A walk is called a path if all vertices are distinct.

4.Let G be a connected graph with vertices $1, 2, ...n$ .The distance d(i,j) between the vertices i and j is defined as the minimum length of an i-j path d(i,j)=0. The maximum value of d(i,j) is the diameter of G.

Matrices

1. An m×n Matrix consist of mn real numbers arranged in m rows and n coloumns.The entry in row i and coloumn j of the matrix A is denoted by a_{ij}.

2. The transpose of an m×n matrix A is an n×m matrix A.

3. Let A be a square matrix of order n.The entries a_{11},a_{22} ,......a_{nn} are said to constitute the diagonal of A.The trace of A is defined as tr(A)=a_{11}+ a_{22} +......+ a_{nn}.

4. Let A be an m×n matrix.The determinant det(λI-A)is a polynomial in the (complex) variable λ of degree n and is called the characteristic polynomial of A.
The equation det(λI-A) = 0 is called the characteristic equation of A.

By fundamental theorem of algebra the equation has n complex roots and these roots are called the eigen values of A.
The Eigen values might not all be distinct.The number of times an eigen value occurs as root of the characteristic equation is called algebraic multiplicity of an eigen value.
If λ_1 , λ_2 ,......, λ_n are eigen values of A,then we may factor the characteristic polynomial as
det(λI-A) = (λ-λ_1) (λ- λ_2)......(λ - λ $_n$)
also detA = λ_1 , λ_2 ,......, λ_n and tr(A)= λ_1 + λ_2 + ...+ λ_n .

5.The geometric multiplicity of the eigen value of A is defined to be the dimension of the null space of λI-A.

7.If f(A) is a polynomial in A,then eigen values of f(A)are f(λ_1) .....f(λ_n).

8. A square matrix is called symmetric if A=A^T.

9. The eigen values of a symmetric matrix are real also the algebraic and geometric multiplicities of any eigen value coincide.The rank of the matrix equals the number of non-zero eigen values,counting the multiplicities.

10. In graph theory,the girth of a graph is the length of a shortest cycle contained in the graph.If graph doesnot contain any cycles its girth is defined to be infinity.

11. Newtons Method in numerical analysis also known as Newton-Raphson Method is a method for finding successively better approximations to the roots of a real valued function.

12. The positive semidefinite matrix is one that is Hermitian , and whose eigen values are all non-negative.A Hermitian matrix is one which is equal to its conjugate transpose.

CHAPTER 1
BASIC PROPERTIES OF MATRICES AND GRAPH SPECTRA.

This Chapter gives a survey of the relationship between the properties of a graph and the spectrum of its adjacency matrix and laplacian matrix since eigen values are at the heart of understanding the properties and structures of a graph.
A Graph G can be represented in matrix form i.e., adjacency matrix A, it is a square symmetric matrix and all of the elements are non-negative,$a_{ij}=a_{ji}$

Properties of Adjacency Matrix :

- **PROPERTY 1 :** : The number of walks of length l from V_i to V_j in G is the element in position (i,j) of the matrix A^l.

- **PROPERTY 2 :** : The trace of A^2 is twice the number of edges in the graph.

- **PROPERTY 3 :** : The trace of A^3 is six times the number of triangles in the graph.

- **PROPERTY 4 :** : The coefficients of the characteristic polynomial that coincide with matrix A of G have following characteristics ,
1)$C_1 = 0$.
2)$-C_2$ is the number of edges of G.
3)$-C_3$ is twice the number of triangles in G,
where the characteristic polynomial is $\lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2}.....+C_n$.

- **PROPERTY 5 :** : The sum of eigen values of a matrix equals its trace.

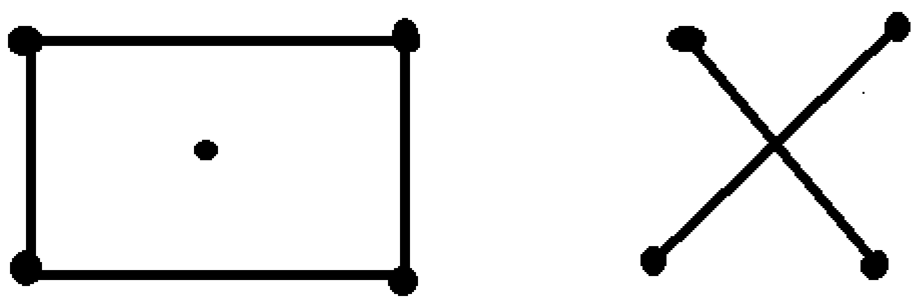
- **PROPERTY 6 :** If a matrix is real symmetric ,then each eigen value of the graph relating to the matrix is real.

• **PROPERTY 7 :** : The geometric and algebraic multiplicity of each eigen value of a real symmetric matrix are equal.

• **PROPERTY 8 :** : The eigen vectors that corresponds to the distinct eigen values are orthogonal

• **PROPERTY 9 :** : If a graph is connected ,the largest eigen value has multiplicity of 1.

NOTE: : If eigen values of 2 graphs donot match then the graphs are not isomorphic.
converse is not true i.e.,graphs which are not isomorphic but have same eigen values.

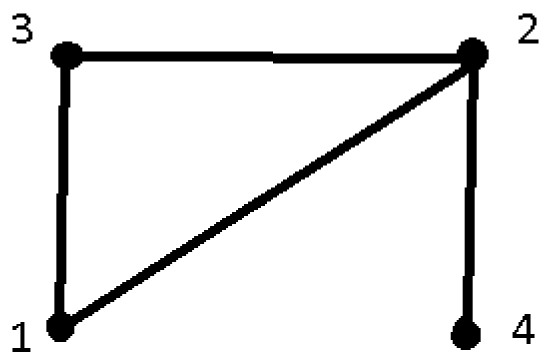


they have eigen values -2,0,0,0,2.
The largest eigen value $\lambda_1(G)$ is called the index of G.

Laplacian Matrix

The laplacian is a alternative to the adjacency matrix for describing adjacent vertices of a graph.It is a square matrix ,the main diagonal of the matrix represents the degree of the vertex while the other entries are as follows,
L_{ij} = {-1 , if V_i and V_j are adjacent}
0r it gives 0 otherwise .
Laplacian can also be derived from D-A , where D is diagonal matrix whose entries represent degrees of the vertices and A is the adjacency matrix .

EXAMPLE :



$$A=\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad D=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D\text{-}A = L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The Laplacian of a connected graph has eigen values $\lambda_1 \leq \lambda_2 \leq ..... \leq \lambda_n$. The algebraic connectivity is defined to be λ_2, the second smallest eigen value.

• **PROPERTY 1 :** The smallest eigen value of L is 0. Laplacian matrix being a semi-definite matrix have n real laplace eigen values $0=\lambda_1 \leq \lambda_2 \leq ..... \leq \lambda_n$.

• **PROPERTY 2 :** The multiplicity of 0 as an eigen value of L is the number of connected components in the graph.

• **PROPERTY 3 :** The algebraic connectivity is positive if and only if the graph is connected.

• **PROPERTY 4 :** The eigen values of a self adjoint matrix are all real. laplacian matrix is a self adjoint matrix.
PROOF : Suppose λ is an eigen value of the self adjoint matrix L and V is a non-zero eigen vector of λ. Then

$\lambda \parallel v \parallel^2 = \lambda < v,v > = < \lambda v , v > = < Lv , v > = < v, Lv > = \overline{\lambda} < v,v >$
so we have $\lambda \parallel v \parallel^2 = \overline{\lambda} \parallel v \parallel^2$
since $v \neq 0$ we have $\parallel v \parallel^2 \neq 0$
i.e., $\lambda = \overline{\lambda}$
so λ is real.
Eigen values of a Laplacian matrix are all real since ,Laplacian Matrix of a graph is symmetric and consist of real entries so $L_G = L_{G^*}$ where L_{G^*} is conjugate transpose of L_G . so, L_G is Self Adjoint. By theorem all eigen values of L_G are real.

• **PROPERTY 5 :** The adjacency eigen values lies in the interval [-d,d] . The laplacian eigen values lies in the interval [0,2d].

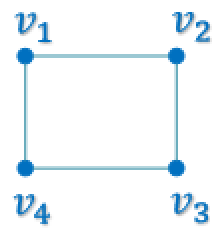
•**PROPERTY 6 :** The number of distinct adjacency respectively laplacian eigen values is atleast $\delta + 1$.

SPECTRUM OF A GRAPH

The spectrum of a graph is the set of eigen values of G together with their algebraic multiplicities or number of times that they occur. If a graph has k distinct eigen values $\lambda_1 > \lambda_2 > ...... > \lambda_k$. with multiplicities $m(\lambda_1)$, $m(\lambda_2)$,...,$m(\lambda_k)$ then the spectrum of G is written as

Spec(G) = $\begin{pmatrix} \lambda_1 & \lambda_2 & . & . & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & . & . & m(\lambda_n) \end{pmatrix}$ where $\sum_{i=0}^{k} \lambda_i = n$

Example: Spectrum



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 & 0 & -1 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & -1 \\ -1 & 0 & -1 & \lambda \end{pmatrix}$$

$$= \lambda^2(\lambda - 2)(\lambda + 2).$$

$$spec(C_4) = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

NOTE : For the adjacency matrix and laplacian matrix following can be deduced from the spectrum:
(1) The number of vertices.
(2)The number of edges.
(3)Whether G is regular.
(4)Whether G is regular with any fixed girth.
For adjacency matrix following follows from spectrum:
(5)The number of closed walk of any fixed length.
(6) Whether G is bipartite.
For laplacian matrix the following follows from spectrum:
(7)The number of components.
(8)The number of spanning trees.

PROOF: (1) We have ,A graph with n vertices have n eigen values .Thus proof is trivial.
(2) and (5) have been proved from the theorem that ,For any n×n matrix A and B,the following are equivalent: • A and B are cospectral.
• A and B have the same characteristic polynomial.
• tr(A^i) = tr(B^i) for i= 1,2,...n.

for,
A pair of graphs are cospectral mates if they have same spectrum but are non-isomorphic.
also by Newtons relations the roots $\lambda_1 \geq \lambda_2 \geq ......\ \lambda_n$ of a polynomial of degree n are determined by the sums of the powers $\sum_{j=1}^{n} {\lambda_j}^i$, i=1,2,...n.
Now tr(A^i) is the sum of the eigen values of A^i which equals the sum of i^{th} powers of the roots of characteristic polynomial.
If A is the adjacency matrix of a graph,then tr(A^i) gives the total number of closed walks of length i, in particular ,they have the same number of edges and triangles(i=3).
so cospectral graphs have same number of closed walks of length i.

Now ,(4) follows from (5) since, G is bipartite iff G has no closed walks of odd length.
(3) follows from preposition which states,
Let $\alpha \neq 0$ with respect to the matrix Q= αA +βJ + γD + δI , a regular graph cannot be cospectral with a non-regular one, except possibly when γ=0 and -1 < $\frac{\beta}{\alpha}$ < 0 .

and(4) follows from (3) and the fact that in a regular graph the number of closed walks of length less than the girth depends on the degree only.
The last two statements follows from the well-known results of laplacian matrix.Indeed the co-rank of L equals the number of components and if G is connected, the product of the non-zero eigen values equals n times the number of spanning trees.

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CHAPTER 2
SPECTRA OF GRAPH CLASSES

In this chapter we compute the spectra of graph classes.It contains several results on the eigen values of graphs.

CIRCULANT MATRIX

A circulant matrix of order n is a square matrix of order n in which all the rows are obtainable by successive cyclic shifts of one of its row (usually first row).

Example

Circulant with first row (a_1, a_2, a_3, a_4) is the matrix, $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_1 \end{bmatrix}$

LEMMA 2.1 :
Let A be a circulant matrix of order n with first row (a_1, a_2,a_3...a_n) then

S_p(a)={a_1+a_2w+a_3w^2+...+a_nw^{n-1}: w is an n^{th} root of unity }
S_p(a)={a$_1$+ρ^r+ρ^{2r}+ρ^{3r}+.........+$\rho^{(n-1)r}$:} 0⩽r⩽n-1 and ρ is a primitive n^{th} root of unity }

PROOF: Given A is the Circulant Matrix of Order n with first row $(a_1, a_2, ..., a_n)$ then $\begin{bmatrix} a_1 & a_2 & \cdots\cdots & a_n \\ a_n & a_1 & \cdots\cdots & a_{n-1} \\ a_{n-1} & a_n & \cdots\cdots & a_{n-2} \\ & \cdots\cdots\cdots\cdots\cdots\cdots & & \\ a_2 & a_3 & \cdots\cdots & a_1 \end{bmatrix}$

Hence, D= $|\lambda I - A| = \begin{bmatrix} \lambda - a_1 & -a_2 \ldots\ldots & -a_n \\ -a_n & \lambda - a_1 \ldots\ldots. & -a_{n-1} \\ & \ldots\ldots\ldots\ldots\ldots & \\ -a_2 & -a_3 \ldots\ldots & \lambda - a_1 \end{bmatrix}$

Let C_i be the i^{th} column of 0 ,1≤ i ≤ n and , w be an n^{th} root of unity.
Replace C_1 by C_1 + C_2w + ... + $C_n w^{n-1}$

Let C_i be the i^{th} column of 0 ,1≤ i ≤ n and , w be an n^{th} root of unity.
Replace C_1 by C_1 + C_2w + ... + $C_n w^{n-1}$

$$D = - \begin{bmatrix} \lambda - a_1 - a_2 w - ... - a_n w^{n-1} & -a_2 & ...... & -a_n \\ w(\lambda - a_1 - a_2 w - ... - a_n w^{n-1}) & \lambda - a_1 & ..... & -a_{n-1} \\ & .... & & \\ w^{n-1}(\lambda - a_1 - a_2 w - ... - a_n w^{n-1}) & -a_3 & ..... & \lambda - a_1 \end{bmatrix}$$

Now taking λ_w =a_1 + a_2w+ ...+ $a_n w^{n-1}$. we have,

$$D = \begin{bmatrix} \lambda - \lambda_w & -a_2 & .... & -a_n \\ w(\lambda - \lambda_w) & \lambda - a_1 & ...... & -a_{n-1} \\ & .... & & \\ w^{n-1}(\lambda - \lambda_w) & -a_3 & .... & \lambda - a_1 \end{bmatrix}$$

Hence $(\lambda - \lambda_w)$ isa factor of D.
This gives D =$\prod_{ww^n=T}$ (λ-λ_w)
Thus $S_p(A) = \{\lambda_w$: w^n=1$\}$.

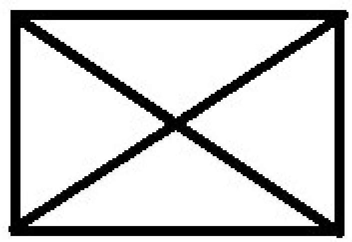
SPECTRUM OF COMPLETE GRAPH K_n :

The spectrum of complete graph K_n can be determined easily . The adjacency Matrix A is given by A= $\begin{bmatrix} 0 & 1 & 1...1 \\ 1 & 0 & 1...1 \\ . & . & ..... \\ 1 & 1 & 1...0 \end{bmatrix}$ which is a circulant matrix of order n with first row (0 ,1, 1,...1) by lemma
λ_w= $a_1 + a_2 w + a_3 w^2 + ... + a_n w^{n-1}$
$= w + w^2 + w^3 + ... + w^{n-1}$
$= \begin{cases} n-1, if w = 1 \\ -1, if w \neq 1 \end{cases}$
where w is the nth root of unity . (1 +w +w^2 +...+ w^{n-1} = 0) for the eigen values n-1 A has eigen vector X = $(1, 1, , ..1)^T$. The matrix $\frac{-1}{n-A}$ has

a rank 1 and thus -1 is an eigen value of multiplicity n-1. Hence n-1 has multiplicity 1.

so, $S_p(\ K_n) = \begin{bmatrix} n-1 & -1 \\ 1 & n-1 \end{bmatrix}$

Example :



Consider K_4

$S_p(K_4)$= $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$

Eigen values of K_4 are 3 and -1 with multiplicities 1 and 3 respectively.

Laplace Matrix is nI-J ,which has spectrum $\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}$ where J is a 1-matrix of order n .

PROPERTY : The complete graph is the only connected graph with exactly two distinct eigen values .

SPECTRUM OF THE CYCLE C_n :

The spectrum of the cycle C_n can be determined by the fact that the adjacency matrix of C_n is a circulant matrix.

Labelling the vertices of the cycle C_n as 0,1,....(n-1) the vetex i is adjacent to i±(mod n).

Hence the adjacency matrix of C_n is given by,

$$A = \begin{bmatrix} 0 & 1 & 0... & 0 & 1 \\ 1 & 0 & 1... & 0 & 0 \\ 0 & 1 & 0... & 0 & 0 \\ . & . & .... & . & . \\ 1 & 0 & 0... & 1 & 0 \end{bmatrix}$$

which is a circulant matrix with first row (0 1 0 ... 0 1)

LEMMA 2.2:

S_pC_n =$\{\rho^r$+$\rho^{r(n-1)}$: $0 \leq r \leq (n-1),\ where\ \rho\ is\ the\ primitive\ n^{th}\ root\ of\ unity\}$
Take $\rho = exp^{\frac{2\pi i}{n}}$=cos $\frac{2\pi}{n}$+ i sin ($\frac{2\pi}{n}$)

$\rho^n + \rho^{r(n-1)}$ =cos$\frac{2\pi r}{n}$ + i sin $\frac{2\pi r}{n}$ +cos $\frac{2\pi r(n-1)}{n}$ +i sin $\frac{2\pi r(n-1)}{n}$

= 2 cos πr cos ($\frac{2\pi r}{n}$ - πr) + i sin πr cos ($\frac{2\pi r}{n} - \pi r$)

= 2 cos ($\frac{2\pi r}{n}$ - πr) $[cos\pi r + i\ sin\pi r]$

= 2 cos $\frac{2\pi r}{n}$. $0 \leq r \leq n-1$

Thus the eigen values C_n are λ,

$\lambda = 2$ cos $\frac{2\pi r}{n}$, $0 \leq r \leq n-1$

But these numbers are not all distinct taking account of coincidences the complete description of spectrum is

$$S_p(C_n) = \begin{pmatrix} 2 & 2cos\frac{2\pi}{n} & ... & 2cos\frac{(n-1)\pi}{n} \\ 1 & 2 & ... & 2 \end{pmatrix}$$

when n is odd

$$= \begin{pmatrix} 2 & 2cos\frac{2\pi}{n} & ... & 2cos\frac{(n-1)\pi}{n} & 2 \\ 1 & 2 & ... & 2 & 1 \end{pmatrix}$$

n is Even.

Examples :

$$S_p(C_6) = \begin{pmatrix} 2 & .618 & -1.618 \\ 1 & 2 & 2 \end{pmatrix}$$

SPECTRUM OF BIPARTITE GRAPHS:

THEOREM 2.3 :
A graph is bipartite if and only if spectrum is symmetric with respect to the origin.

proof: Let G be bipartite graph with bipartition U and V .

Let G has an adjacency matrix of the form, A= $\begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}$ where , Q=P^T

Note that the non-zero entries of P and Q corresponds to the edges incident with the vertices from U and V respectively.

Suppose λ is an eigen value of G and X = $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigen value of G corresponding to λ.

$$\begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} Px_1 \\ Qx_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

thus, P$x_2 = \lambda\ x_1$, Q$x_1 = \lambda\ x_2$.

Now consider vector y= $\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$

Ay = $\begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} -Px_2 \\ Qx_1 \end{pmatrix} = \lambda \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$
i.e., Ay = -λy.

-λ is also an eigen value of G .
So, the spectrum of a bipartite graph is symmetric about 0.
converse is also true.

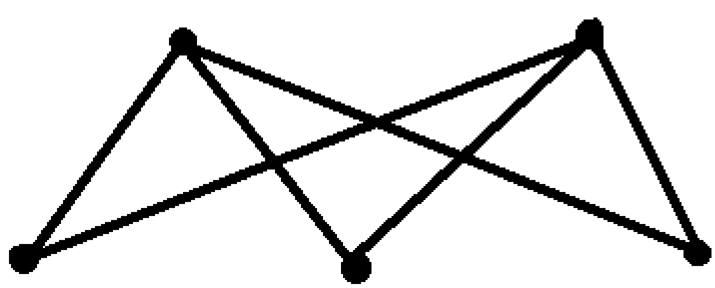
NOTE : A Graph G is bipartite iff λ_1= -λ_n ,also if λ is an eigen value then so is -λ and $C_(2k-1)$=0 for n≥1.

SPECTRUM OF COMPLETE BIPARTITE GRAPH :

THEOREM 2.4 :
Consider the complete bipartite graph ,$K_{p,q}$ then
S_p $(K_{p,q})$ = $\begin{pmatrix} 0 & \sqrt{pq} & -\sqrt{pq} \\ p+q-2 & 1 & 1 \end{pmatrix}$

PROOF : Let V($K_{p,q}$) have the partition (X , Y) with | X | = p and | Y |= q
The adjacency matrix of $K_{p,q}$ is given by, A = $\begin{pmatrix} 0 & J_{p,q} \\ J_{p,q} & 0 \end{pmatrix}$ where $J_{r,s}$ stands for all 1 matrix of size r×s

EXAMPLE : Consider$K_{2,3}$



S_p $(K_{2,3})$ = $\begin{pmatrix} 0 & \sqrt{6} & \sqrt{-6} \\ 3 & 1 & 1 \end{pmatrix}$.

The Laplace spectrum is $\begin{pmatrix} 0 & m & n & m+n \\ 1 & n-1 & m-1 & 1 \end{pmatrix}$

SPECTRUM OF REGULAR GRAPHS:

Following theorem shows that regularity of a graph G is an eigen value of G.
THEOREM 2.5:

Let G be a k-regular graph of order n.then, (1) k is an eigen value of G.
(2)If G is connected every eigen vector corresponding to the eigen value k is a multiple of 1 and the multiplicity of k as an eigen value of G is 1.
(3) For any eigen value λ of G, | λ | $\leq$ k , Hence S_p (G) $\leq$ [-k,k] ,k=degree.

SPECTRUM OF STRONGLY REGULAR GRAPH :

DEFINITION :A Strongly regular graph with parameters (n,k,s,t) is a k regular graph with following properties :
(1) Any two adjacent vertices of G have exactly s common neighbours in G.
(2)Any two non-adjacent vertices of G have exactly t common neighbours in G.
It is usually denoted as srg(n,k,s,t).

EXAMPLE :• The Cycle C_5 is an srg(5,2,0,1).
In C_5 ,any two adjacent vertices have no common neighbours and any two non-adjacent vertices have 1 common neighbours .
•Shrikande graph , S=srg(16,6,2,2).

THEOREM 2.6 :
If G is a strongly regular graph with parameters (n,k,s,t) then k(k-s-1) = t(n-k-1).
PROOF: We prove this theorem by counting the number of induced paths on three vertices in G having the same vertex V of G as end vertex in two different ways. Consider the vertex V of G. There are k neighbours W of V in G.since G is a strongly regular graph, for each W, there are s vertices that are common neighbours of V and W.

The remaining k-1-s neighbours of W induces a P_3 with V as an end vertex. As this is true for each neighbour W of V in G,there are k(k-s-1) paths of length 2 with V as an end vertex.
so,we can compute this in another way.
There are n-k-1 vertices U of G that are non-adjacent to V.V and U have t common neighbours. Each one of these t vertices give rise to an induced P_3 in G with V as an end vertex.
The number is t(n-1-k). so we have, k(k-s-1) = t(n-1-k).

THEOREM 2.7 :
Let G be srg(n,k,s,t) and let A be the adjacency matrix of G ,then A^2 = kI + sA + t(J-I-A).

LEMMA 2.8 :
Let G be a graph which is neither complete nor empty and let A be the adjacency matrix of G . Then G is strongly regular if A^2 is a linear combination of A,J and I.
PROOF : Let A be the adjacency matrix of G.
Given A^2 is a linear combination of A ,J and I.
Let A^2 = aA + bJ + cI where a,b,c are scalars. for i=j , the diagonal entries in A^2 will be b+c.
By Lemma which states The number of walks of length l in G from V_i to V_j is the entry in position (i,j) of the matrix A^l. Thus there are b+c walks from V_i to V_j so each vertex of G is adjacent to b+c vertices. Hence G is b+c regular.
If $U_iV_j \in$ E(G) ,then the $(i,j)^th$ entry of A^2 will be (a+b). so by Lemma, there are a+b walks of length 2 from V_i to V_j .Thus any two adjacent vertices in G have a+b common neighbours.
Also if V_i V_j is not an edge of G,then the $(i,j)^th$ entry of A^2 will be b. Hence there are b walks in G from V_i V_j .By Lemma two non adjacent vertices in G have b common neighbours.
Thus G = srg(n, b+c, a+b, b).

THEOREM 2.9 :
Let G be a strongly regular graph with parameters (n,k,s,t) and let A be the adjacency matrix of G. Let Δ=$(s-t)^2$ + 4(k-t).Then any eigen value of A is either k or $\frac{1}{2}$ (s-t$\pm$ $\sqrt{\Delta}$) .

PROOF : We know K is an eigen value of A with 1 as the multiplicity and 1 as the corresponding eigen vectors since G is k-regular.
Let λ be another eigen value of A with y as the corresponding eigen vector so that Ay=λy.
By Lemma, A^2 = kI + sA + t(J-I-A).
A^2y = ky + sAy + t(-y-Ay)
λ^2y = ky + sλy + t(-y-λy)
λ^2 = k + sλ + t(-1-λ)
λ^2 - λ(s-t)-(k-t) =0
so we have $\lambda = \frac{1}{2}$ (s-t$\pm$ $\sqrt{(s-t)^2 + 4(k-t)}$
= $\frac{1}{2}$ (s-t$\pm$ $\sqrt{\Delta}$) .
Thus the eigen values of G must be either k or $\frac{1}{2}$ (s-t$\pm$ $\sqrt{\Delta}$) .

THEOREM 2.10 :
Let G be a connected strongly regular graph with parameters (n,k,s,t). Let Δ=$(s-t)^2$ + 4(k-t) and b=n-k-1. Then the number $m_1 = \frac{1}{2}$(n-1 + $\frac{(n-1)(t-s)-2k}{\sqrt{\Delta}}$) and $m_2 = \frac{1}{2}$(n-1 - $\frac{(n-1)(t-s)-2k}{\sqrt{\Delta}}$) are non-negative integers.

EXAMPLE : The spectrum of the shrikande graphs,S=srg(16,6,2,2) is given by the theorem 2.8 and 2.9.
Δ =$(s-t)^2$ + 4(k-t)= 0 + 4(6-2) = 16

$\lambda = \frac{1}{2}$ (s-t$\pm$ $\sqrt{\Delta}$) = $\frac{1}{2}$ (2-2 $\pm$ $\sqrt{16}$) = $\pm$2 are the eigen values.

Thus λ_1 = 2 , λ_2 = -2 .

$m_1 = \frac{1}{2}$(n-1 + $\frac{(n-1)(t-s)-2k}{\sqrt{\Delta}}$) = $\frac{1}{2}$ (15-3) =6 ,

$m_2 = \frac{1}{2}$(n-1 - $\frac{(n-1)(t-s)-2k}{\sqrt{\Delta}}$)

are the multiplicities of λ_1 and λ_2.
k=6 is also an eigen value with multiplicity 1 as it is connected.

$S_p(S)$= $\begin{pmatrix} 6 & 2 & -2 \\ 1 & 6 & 9 \end{pmatrix}$.

THEOREM 2.11 :
Let G be a connected regular graph with exactly three distinct eigen values. Then G is strongly regular.
PROOF : Let G have n vertices and suppose it is k-regular.since G has 3 distinct eigen values, it has diameter atmost 2 by a corollary since G has 3 distinct eigen values,it has diameter atmost 2 by a corollary ,Let G be a connected graph with k distinct eigen values and let d be the diameter of G.Then k $>$ d .Since,G is connected and is neither complete nor empty,its diameter cannot be 0 or 1 and hence it must be 2. k is an eigen value of G since it is k-regular. Let λ_1 and λ_2 be the other two eigen values and let $P(\lambda) = (\lambda\text{-}\lambda_1)(\lambda\text{-}\lambda_2)$.
Then (A-KI)P(A) = 0 .
since G is connected ,k has multiplicity 1 and hence the null space of (A-KI) is spanned by 1. As (A-KI)P(A) =0 each column of P(A) is a multiple of 1.Also since P(A) is symmetric. P(A) = αJ for some α.
(A-λ_1 (A-λ_2)) =αJ
Thus A^2 is a linear combination of A,I and J.
By Lemma 2.7 G is strongly regular graph.

CHAPTER - 4
APPLICATION OF EIGEN VALUES OF GRAPHS

Eigen values of graphs appear in mathematics ,physics, chemistry and computer science . It is often efficient in counting structures i.e., acyclic digraphs,spanning graphs, Hamiltonian cycles, independent sets, K colourings etc.

• EIGEN VALUES IN APPLIED SCIENCE:

1. INFORMATION TECHNOLOGY:

In shanon IT,the channel capacity which characterizes the maximum amount of information that is transmitted over a channel or stored into a storage medium per bit can be expressed in terms of the eigen values of its channel graph. Combinatorically,the capacity can be discussed by counting the number of closed walks of length k in the channel graph G and then by letting the k tend to infinity .
construction of encoder or decoder for a given code is based on the largest eigen value of its channel graph.

2. In QUANTUM CHEMISTRY :

Here ,the skeleton of a non-saturated hydrocarbon is represented by a graph. The energy levels of the electrons in such a molecule are the eigen values of the graph.

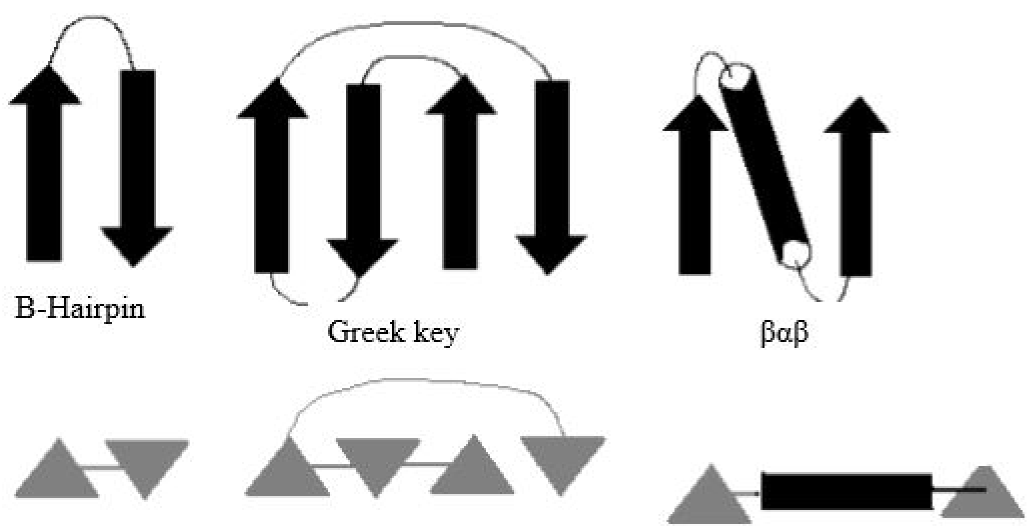
The stability of molecule is closely related to the spectrum of its graph .
correspondence :
$Vertex - CarbonAtom$
$Edge - Bond$
$Vertexdegree - Valency$
Adjacency Matrix -Topological Matrix

3. PROTIEN STRUCTURE:

The 3 dimensional stucture of protien is the key to understand their function and evolution$[V_i, P_10]$.

A Protien is formed by a sequential joining of amino acids end-to-end to form along chain like molecule or polymer called polypeptides. Four major protien class are shown in figure
Cylinder represent helices and arrows represent strands.Two challenges are identifying the fold adopted by polypeptide chain and identifying similarities in protien structure.

Graphs have helped to represent the topology of protien structures no matter how complex.The main problem is to define vertices and edges. Below are some examples of protien with their graphs below,



The basic unit of a protien is its amino acids residue:
- To study cluster identification the amino acids represent the vertices and three dimensional connectivity between them is represented by edges.
-To study fold and pattern identifiaction and the folding rules of proteins, $\alpha - helices$ and β strands are used for vertices and spatially closed structures are usedfor edges.

-To identify protein with similar folds the backbones are the vertices and spatial neighbours within a certain radius are edges.
- Connected graphs are used to represent αhelical structures.The vertices represent secondary structures and edges represent contact between helices. That is, Properties of graphs and their graph spectral give information about protein structures.

4.IN PHYSICS :
Treating the memmbrane vibration problem by approximative solving of the corresponding partial differential equation leads to consideration of eigen values of a graph which is a discrete model of the membrane.
The spectra of graphs appear in a number of problems in statistical physics for example,we mention ,dimmer problem.
The dimmer problem is related to the investigation of the thermodynamic properties of a system of diatomic molecules,absorbed on the surface of a crystal.The most favourable points for the absorption of atoms on such a surface form a 2-dimensional lattice and a dimmer can occupy two neighbouring points.It is necessary to count all ways in which dimers can be arranged on the lattice without overlapping each other, so that every lattice is occupied.
A graph can be associated with a given absorption surface.The vertices of the graph represent the points which are the most favourable for absorption.Two vertices are adjacent if and only if the corresponding points can be occupied by a dimmer .
In this manner an arrangement of dimmers on the surface determines a 1-factor in the corresponding graph or viceversa.
Thus dimmer problem is reduced to the task of determining the number of 1-factor in a graph .

- **IN GRAPH COLORING:**

One of the classic problems in graph theory is vertex-coloring, which is the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices have the same color. The object is to use as few colors as possible.

The proper coloring of a graph also forms a natural partition of the vertex set of a graph. The chromatic number, $\chi(G)$, is the least number of colors required for such a partition. A graph G is l-critical if $\chi(G) = l$ and for all induced subgraphs $A \neq G$ we have $\chi(A) < l$. The spectrum of a graph gives us insight into the chromatic number.

We have a property 1, that , Given a graph G with χ (G) = l $\geq$ 2, there exists a subgraph of G, A$\neq$ G, such that χ (A)= l, and every vertex of A has degree $\geq$ l in A.

Property 2 says , For any graph G, $\chi(G) \leq 1 + \lambda_1$, where λ_1 is the largest eigenvalue of the adjacency matrix of G.

Proof: From Property 1, there is an induced subgraph A of G such that $\chi(G) = \chi(A)$ and $dmin(A) \geq \chi(G) - 1$, where dmin(A) is the least degree of the vertices of A.

Thus we have $\chi(G) \leq 1 + dmin(A) \leq 1 + \lambda_1(A) \leq 1 + \lambda_1(G)$.

Of course, the absolute largest value of $\chi(G)$ is n, the number of vertices. If $\lambda_1 < n - 1$,

then we will have a smaller maximum than n.

Property 3: The lower bound of the chromatic number is

$\chi(G) \geq$ 1+ $\frac{\lambda_1}{-\lambda_{min}}$

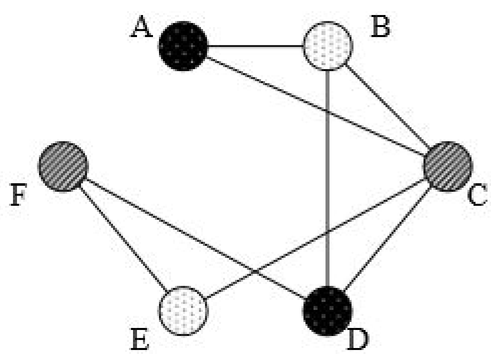
A classical application of vertex coloring is the coloring of a map. To make the process as inexpensive as possible, we want to use a few colors as possible. In the graph, the countries are represented by vertices with edges drawn between those countries that are adjacent. Determining the chromatic number of the graph gives us the fewest colors necessary to color the map.

Another application to graph coloring is a sorting problem, such as sorting fish in a pet store. Some fish can be in the same tank together, while other fish cannot. Say we have fish types A, B, C, D, E and F. They can be put into tanks according to the chart below

Type of Fish	A	B	C	D	E	F
Cannot be with fish type(s)	B, C	A, C, D	A, B, D, E	B, C, F	C, F	D, E

If the graph is set up such that the fish are the vertices and edges are drawn between those that cannot be in a tank together, the chromatic number will tell us how many tanks we need.

G =



The adjacency matrix for G is $A = \begin{Bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{Bmatrix}$ which has $\lambda_1 =$ 2.853 and $\lambda_(min) = 2.158$ Substituting these into the formula from Property 4-3, $\chi(G) \geq 1 + \frac{\lambda_1}{-\lambda_{min}}$

we get $\chi(G) \geq 2.322$. This tells us that we need at least three colors for our graph. This will save us time by preventing us from attempting to color it with fewer than 3 colors. In this case, we will need 3 tanks, as shown above in Figure . One tank will hold fish A and D, the second tank will hold fish B and E, and the third tank will hold fish C and F. There are many applications of graph coloring, and spectral of the graph gives us insight to the chromatic number.

- **IDENTIFYING CLUSTERS :** is an important aspect in the field of electrical network connections[$Vi, p9$]. Graph spectral method is extremely helpful in this, and can find the needed results with minimal computations. An adjacency matrix is used, but edge weights are used as entries.The weights are $\frac{1}{d_{ij}}$.
where $d_i j$ represents the distance from vertices i and j. The goal is to find the location of n vertices that minimizes the weighted sum of the squared distances between the vertices.
The key point of interest is that the second smallest eigenvalue of the Laplacian matrix, λ_2, and its vector component gives the clustering points in the graph. The vertices that are clustered have the same value for the second smallest eigenvalue. Also, the largest eigenvalue contains information regarding only one of the clusters. The vertex with the largest vector component is the vertex with the highest degree.

- **IN SOCIAL SCIENCE :**
Social network have been studied actively in social sciences ,where the general feature is that the network are viewed as static graphs whose vertices are individuals and whose edges are the social interactions between these individuals . The problem is to analyze the topology and dynamics of data sets which have relationships between themselves in the network.

FRIENDSHIP THEOREM :

As an application of theorem 2.11 we prove Friendship Theorem which can be stated as follows; Suppose in a group of atleast 3 people we have the situation that any pair of persons have precisely one common friend, then there is always aperson who is everybodys friend.

THEOREM 3.1: Let G be garph in which any two distinct vertices have exactly one common neighbour .Then G has a vertex that is adjacent to every other vertex and more precisely G consist of a number of triangles with a common vertex.

PROOF : From the hypothesis ,it easily follows that G is connected.Let the vertices of G be $1, 2, ...n$.
Let i and j be non adjacent vertices of G, Let $N(i)$ and $N(j)$ be their respective neighbour sets.with u $\in N(i)$, we associate that v$\in N(j)$ which is unique common neighbour of u and j.
set v=f(u) and f is one to one mapping from N(i) to N(j) .Indeed if w $\in N(i)$,w $\neq$ u satisfies f(w)=v then u and w would have two common neighbours namely i and v which is a contradiction to our hypothesis.
Therefore,f is one-one and hence $|N(i)| \leq |N(j)|$.
similarly $|N(j)| \leq |N(i)|$.
i.e., $|N(i)| = |N(j)|$.
Suppose G is k-regular ,by the hypothesis G must be strongly regular with parameters (n,k,1,1). Then
$\Delta = 4(k-1)$ and so, $\sqrt{\Delta} = 2\sqrt{k-1}$ also

$m_1 = \frac{1}{2}(n-1-\frac{2k}{2\sqrt{k-1}})$ and $m_2 = \frac{1}{2}(n-1+\frac{2k}{2\sqrt{k-1}})$

$m_1 - m_2 = \frac{1}{2}(n-1-\frac{2k}{2\sqrt{k-1}})$

is an integer by theorem 2.9.
so k divides k^2 which happens only when k=0 or k=2.
If k=0 ,G is connected n=1 ,then the theorem holds trivially.
If k=2 then in view of hypothesis that any two vertices have exactly one common neighbour,G must be the complete graph on 3 vertices and again theorem holds.
Now suppose G is not regular,then there must be adjacent vertices i and j with unequal degrees.
Let u be the unique common neighbour of i and j and assume without loss of generality that degrees of i and u are unequal.
Let v be any vertex other than i,j and u .If v is not adjacent to both i and j,then the degree of i and j would be equal to that of v ,which is not possible.
Hence v is adjacent to either i and j similarly,v is adjacent to either i or j .similarly v is adjacent to either i or u.

since v cannot be adjacent to both j and u then v must be adjacent to i. Therefore all the vertices other than i and j are adjacent to i . Thus G consist of a number of triangles with i as the common vertex.

REMARK : Thus if any two individuals in a group have exactly one common friend,then there must be a person who is a friend of everybody. This justifies the name ”FRIENDSHIP THEOREM ”.

CONCLUSION

We consider the question to what extend graphs are determined by their Spectrum.In this paper we introduce some basic ideas of spectral graph theory ,primarily focusing on finding the spectra of certain types of graphs and its algebraic connectivity.
Also we found that there is a wide range of application for eigen values of graphs and hence the spectra of graphs .
The research in this field changes its direction to further studies of characterizations of Spectrum and expand to other matrices i.e., Signless Laplacian Matrix and Normal Laplacian Matrix.
I hope I made a basic idea to stimulate more research in this field.

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