A STUDY ON REPRESENTATION THEORY OF FINITE GROUPS

A Dissertation submitted in partial fulfillment of the

Requirementfor the award of

DEGREE OF MASTER OF SCIENCE

IN MATHEMATICS

By

ANILA J CHANDRAN

REGISTER NO: SM16MAT003

(2016 – 2018)

DEPARTMENT OF MATHEMATICS

ST.TERESA'S COLLEGE, (AUTONOMOUS)

ERNAKULAM, KOCHI - 682011

APRIL 2018

DEPARTMENT OF MATHEMATICS

ST.TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM

CERTIFICATE

This is to certify that the dissertation entitled "A STUDY ON REPRESENTATION THEORY OF FINITE GROUPS" is a bonafide record of the work done by ANILA JCHANDRANunder my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College(Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

Anna TreesaRaj (Supervisor)

Assistant Professor

Department Of Mathematics

St.Teresa's College,(Autonomous)

Ernakulam

Smt Teresa Felitia (HOD)

Associate Professor

Department of Mathematics

St. Teresa's College (Autonomous)

Ernakulam

External Examiners:

1. ....2. ....

DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Anna Treesa Raj, Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

ERNAKULAM

APRIL 2018

ANILA J CHANDRAN

REGISTER NO:SM16MAT003

ACKNOWLEDGEMENT

I express my sincere gratitude to my guide Ms.AnnaTreesa Raj, Assistant Professor, Department of Mathematics, St.Teresas College, Ernakulam , for helping me in the successful completion of this work.

I further express my gratitude to Smt.TeresaFelitia, Head of department of Mathematics, St.Teresas College, Ernakulam and other teachers of the Department for their encouragement, suggestions and assistance in taking up this dissertion.

I finally thank my parents, friends and all my well wishers who supported me during the preparation of this project. I bow my head before God almighty without whose blessings I wouldn't be able to complete this work.

ERNAKULAM

APRIL 2018

ANILA J CHANDRAN

REGISTER NO: SM16MAT003

INTRODUCTION

Representation theory is a branch of Mathematics that studies abstract algebraic structures by representing their elements as linear tranformation of vector spacec and studies modules over these abstract algebraic structures.A representation makes an abstract algebraic object more concrete by describing its elements by matrices and the algebraic operations in terms of matrix addition and matrix multiplication.The algebraic objects amenable to such a description include groups,associative algebras and Lie algebras.The most prominent of these is the Representation theory of groups in which elements of a group are represented by invertible matrices in such a way that the group operation is matrix multiplication.

Representation theory is a useful method because it reduces problems in Absrtact Algebra to problems in Linear Algebra,a subject that is well understood.Furthermore,the Vector Space on which a group is represented can be infinite dimensional and by allowing it to be, for instance,a Hilbert space,methods of analysis can be applied in the theory of groups.Representation theory is also important in Physics because,for eg,it describes how the symmetry group of a physical system affects the solutions of equations describing that system.

Representation theory was born in 1896 in the work of the German Mathematician F.G Frobenius.The work was triggered by a letter to Frobenius by R.Dedikind.In this letter Dedikind made the following observation:
Take the multiplication table of a finite group G and turn it into a matrix X by replacing every entry g of this table by a variable Xg.Then the determinant of X factors into a product of irreducible polynomials{Xg} in ,each of which occurs with multiplicity equal to its degree.Dedikind checked this surprising fact in a few special cases,but could not prove it in general.So he gave this problem to Frobenius.In order to solve this problem,Frobenius created Representation Theory of Finite Groups.

A feature of Representation Theory is its pervasiveness in Mathematics.There are two sides to this.First,the application of Representation Theory are diverse in addition to its impact on Algebra,Representation theory illuminates and generalizes Fourier analysis via Harmonic analysis is connected to Geometry via invariant theory and the Erlangen program and has an impact in Number Theory via automorphic forms and the Langlands program.

The second aspect is the diversity of approaches to Representation Theory.The same objects can be studied using methods from Algebraic Geometry,Module Theory,Analytic Number Theory,Differential Geometry,Operator Theory,Algebraic combinatorics and Topology.

The sucess of Representation Theory has led to numerous generalizations.One of the most general is in Character Theory.The algebraic objects to which Representation Theory applies can be viewed as particular kinds of categories and the representations as functions from the object category to the category of Vector Spaces.This description points to two obvious generelizations:first,the algebraic objects can be replaced by more general categories;second,the target category of Vector Spaces can be replaced by other well understood categories.

PRELIMINARIES

VECTOR SPACE:

A vector space over a field F is a set V together with two operations'+'and '.' are defined, called vector addition and vector multiplication, such that $\forall$ u,v,w in V and c,d be scalars

Associative law of addition : u+(v+w)=(u+v)+w

Commutative law of addition : u+v=v+u

Additive identity : $\exists$ an element 0 in V called the zero vector such that 0+v=v+0=v

Additive inverse : $\forall$v in V $\exists$ an element -v in V,called the additive inverse of v such that v+(-v)=0

Associative law of multiplication : c.(u.v) = (c.u).v

Unitary law : 1.v = v, 1 is the multiplicative identity

Distributive law : c.(u+v) = c.u + c.v

Distributive law : (c+d).v = c.v +d.v

LINEAR COMBINATION:

Let V be a vector space.If $v_1, v_2, ...v_n$ $\in$V,then any vector V=$c_1v_1 + c_2v_2 + ... + c_nv_n$ where$c_1, c_2, ..., c_n$$\in$ F is called a linear combination of the vectors $v_1, v_2, ..., v_n$.

LINEAR SPAN:

Let V be a Vector Space and S be any non-empty subset of V,then the linear span of S is the set of linear combination of finite sets of elements of

S = $\{V_1, V_2, ...V_n\}$ and is denoted by L(S).

LINEARLY INDEPENDENT:

Let V be a Vector Space.A finite set $\{V_1, V_2, ...V_n\}$ of vectors of V is said to be linearly independent if every relation of the form $C_1V_1+C_2V_2+......+C_nVn = 0$,$C_i \in$ F where i = 1,2,....n such that

$C_1 = C_2 = ... = C_n = 0$

LINEARLY DEPENDENT:

Let V be a Vector Space.A finite set $\{V_1, V_2, ...V_n\}$ of vectors V is said to be linearly dependent if $\exists$ scalars $C_1, C_2, ...C_n \in$ F not all of them 0 such that

$C_1V_1 + C_2V_2 + ... + C_nV_n = 0$

BASIS:

A subset S of a Vector Space V is said to be a basis for V if

- S consists of linearly independent vectors.
- Each vector in V is a linear combination of a finite number of elements of S.

<u>LINEAR TRANSFORMATION:</u>

Let U and V be two vector spaces.Then a mapping f :U$\longrightarrow$V is called a Linear Transformation of U into V if

- f $(U_1 + U_2)$ = f(U_1) + f(U_2) ;$\forall$ U_1 , U_2 $\in$ U

- f(cU) = c* f(U) ; $\forall$ u $\in$ U , $\forall$ c $\in$ F.

<u>GROUP:</u>

Let G be a non-empty set with binary operation denoted by *.Then the algebraic structure (G*) is a group if the binary operation * satisfies the following postulates.

- Closure property:

a $*$ b $\in$ G $\forall$ a,b $\in$ G.

- Associative property:

(a$*$b) $*$ c = a $*$ (b$*$ c).

- Existence of identity :

There exist an element e $\in$ G such that

e $*$ a = a $*$ e = a $\forall$ a $\in$ G.

- Existence of inverse:

For each element a $\in$ G $\exists$ an element b $\in$ G such that

b $*$ a = e = a $*$ b.

SUBGROUP:

A non-empty subset H of a group G is said to be a subgroup of G if the composition in G is also a composition in H and for this composition H itself is a groip. If H is a subgroup of G , we shall write $H \leq G$.

ADDITIVE GROUP:

An additive group is a group where the operation is caled addition and denoted by +. In an additive group , the identity element is called zero and the inverse of element a is denoted by -a.

ABELIAN GROUP:

A group (G , $*$) is called an abelian group if

$a * b = b * a \; \forall \; a , b \in G$.

GENERALIZING SET OF A GROUP:

Generalizing set of a group is a subset such that every element of the group can be expressed as the combination of finitely many elements of the subset and their inverses.

FINITE GROUP:

If (G, $*$) is a group such that the number of elements of G is finite , then the group (G , $*$) is said to be a finite group and the number of elements of G is called the order of the group G.If G is not finite , then the group (G , $*$) is said to be an infinite group.

GROUP HOMOMORPHISM:

A homomorphism from a group G to a group H is a mapping ϕ : G $\rightarrow$ H that preserves the group operation

$\phi(ab) = \phi(a)\ \phi(b) \; \forall \; a , b \in G$.

GROUP ISOMORPHISM:

Let G and H be two groups and let ϕ : G $\rightarrow$ H be a function.Then ϕ is said to be a group homomorphism if

- ϕ is one - one and onto

- ϕ(ab) = ϕ(a) ϕ(b) $\forall$ a , b $\in$ G.

ENDOMORPHISM:

An endomorphism of a vector space V is a linear map f:V $\rightarrow$ V and an endomorphism of a group G is a group homomorphism f : G $\rightarrow$ G.

Chapter 1

REPRESENTATION OF A GROUP

DEFINITION 1.1:

Let G be a finite group and K be a field.A representation of G over K is a homomorphism ρ :G $\rightarrow$ GL(V) where V is a vector space of finite dimension over field K.The vector space V is called a representationspace of G and its dimension the dimension of representation.

NOTE:

Strictly speaking the pair (ρ , V) is called the representation of G over the field K.But we simply call ρ a representation or V a representation of G.Let us fix a basis $V_1, V_2, ...V_n$ of V.Then each ρ(g) can be written in a matrix form with respect to the basis.This defines a map ρ :G $\rightarrow$GL$_n$(K) which is a group homomorphism.

DEFINITION 1.2:

Let ρ be a representation of G and W $\subset$ V be a subspace.The space W is called a G-invariant (or stable) subspace if ρ (g)(w) $\in$ W $\forall$ w $\in$ W and $\forall$ g $\in$ G.

NOTE:

If W is a G-invariant subspace of V then we restrict the representation to the subspace W and we can define another representation ρ_w : G $\rightarrow$ GL(W) where ρ W (g) = ρ(g)|W. Hence W is also called a subrepresentation.

EXAMPLES:

1. Let G = Z|mZ and K = R.Let V = R^2 with basis e_1, e_2. Then we have representation of Z|mZ is

$$\rho_r : 1 \rightarrow \begin{bmatrix} \cos\frac{2\pi r}{m} & -sin\frac{2\pi r}{m} \\ \sin\frac{2\pi r}{m} & cos\frac{2\pi r}{m} \end{bmatrix} \text{ where } 1\leq \text{ r } \leq \text{ m-1.}$$

Note that there are m distinct representations.

2. Let ϕ : g $\rightarrow$ H be a group homomorphism.Let ρ be a representation of H. Then $\rho \circ \phi$ is a representation.

3. Permutation representation of S_n:

Let S_n be the symmetric group on n symbols and K be any field. Let V = K^n with standard basis $e_1, e_2, ...e_n$.The representation of S_n is defined as

$\sigma(e_i)$ = eσ for $\sigma \in S_n$.

4.Group Action:

Let G be a group and K be a field.Let G be acting on a finite set X ,that is , G $\times$ X $\rightarrow$ X.We denote K[X]= f | f: X $\rightarrow$ K , set of all maps.Then K[X] is a vector space of dimension | X |.

The elements e_x : X $\to$ K from a basis of K[X]. The action give rise to a representation of G on the space K[X], as follows: ρ : G $\to$ GL(K[X]) given by

ρ(g)(f) (x) = f(g^{-1}x) for x $\in$ X.

DEFINITION 1.3:

Let G be a group and K be a field.Let V be a vector space over K.Then ρ (g) = 1 for all g $\in$ G is a representation.This is called trivial representation.In this case , every subspace of V ia an invariant subspace.

DEFINITION 1.4:

REGULAR REPRESENTATION:

Let G be a group of order n and K be a field.Let V = K[G] be an n dimensional vector space with basis as elements of the group itself.We define L: G $\to$ GL(K[G]) by L(g)(h) = gh called the left regular representation.Also , R(g)(h) defines right regular representation of G.

DEFINITION 1.5:

EQUIVALENCE OF REPRESENTATIONS:

Let (ρ , V) and (ρ' , V$'$) be two representations of G.The representations (ρ , V) and (ρ' , V$'$) are called G - equivalent or equivalent if there exist a linear isomorphism T : V $\to$ V' such that

ρ'(g) = Tρ(g)T^{-1} $\forall$ g $\in$ G.

NOTE:

Let ρ be a representation.Fix a basis say, e_1, e_2.......e_n.Then ρ gives rise to a map G $\to$ GL_n(k) which is a group homorphism.If we change the basis of V, then we get a different map for the same ρ.

DEFINITION 1.6:

COMMUTATOR GROUP:

Let G be a group.Consider the set of elements $\{xyx^{-1}y^{-1}| \ x,y \in G\}$ and $G^{=1}$be the subgroup generated by this subset.This subgroup is called the commutator subgroup of G.

DEFINITION 1.7:

ONE DIMENSIONAL REPRESENTATION:

Let G be the set of all one-dimensional representation of G over C.That is , the set of all group homomorphism from G to C^*.

For $x_1, x_2 \in$ G , we define multiplication by

(x_1x_2) (g) = x_1(g) x_2(g)

RESULTS:

1. The trivial representation is irreducible iff it is one-dimensional.

2. One-dimensional representation is always irreducible.

3. If $|$ G $| \geq 2$, then the regular representation is not irreducible.

4. G is an abelian group.

Chapter 2

MASCHKE'S THEOREM

DEFINITION 2.1

IRREDUCIBLE REPRESENTATION:

A representation (ρ , V) is called irreducible if it has no proper invariant subspace , ie, only invariant subspaces are 0 and V.

NOTE:

Let (ρ , V) and ($\rho^{'}$,$V^{'}$) be two representations of G over the field K.Then the direct sum of these two representations ($\rho \oplus \rho^{'}$, $V \oplus V^{'}$) is defined as follows:

$\rho \oplus \rho^{'}$: GL ($V \oplus V^{'}$) such that

$(\rho \oplus \rho^{'})$ g(V , $V^{'}$) = (ρ (g)(v) , $\rho^{'}$ (g) ($V^{'}$))

In the matrix notation , if we have two representations ρ :G $\rightarrow$ GL_n(K) and $\rho^{'}$: G $\rightarrow$ GL_m(K) , then $\rho \oplus \rho^{'}$ is given by

$$g \rightarrow \begin{bmatrix} p(g) & 0 \\ 0 & p'(g) \end{bmatrix}$$

DEFINITION 2.2:

COMPLETELY REDUCIBLE:

A representation (ρ , V) is called completely reducible if it is a direct sum of irreducible ones.Equivalently, V = $W_1 \oplus$ ....$\oplus$ W_n , where W_i is G-invariant irreducible representation.

DEFINITION 2.3:

An endomorphism π : V $\rightarrow$ V is called a projection if $\pi^2 = \pi$.

LEMMA 1:

Let π be an endomorphism.Then π is a projection iff there exists a decomposition V = W $\oplus$ W' such that π(w) = 0 and π (w') = W' and π restricted to W' is identity.

PROOF:

Let π : V $\rightarrow$ V such that π (W , W') = W'.

Then clearly $\pi^2 = \pi$.

$\Rightarrow$ π is a projection.

Suppose that π is a projection.

We claim that V = ker (π) $\oplus$ img (π).

Let x $\in$ ker (π) $\cap$ img (π).Then $\exists$ y $\in$ V such that π (y) = x.

x = π(y) = π^2 (y) (since π is a projection)

= $\pi(\pi$(x)) = π (0) = 0.

therefore , ker $(\pi) \cap$ img $(\pi) = 0$.

Let v $\in$ V.Then v = (v - π(v)) + π(v)

We have π(v) $\in$ im(π) and v - $\pi(v)$ $\in$ ker (π)

since π (v - π(v)) = π(v) - π^2(v) = π(v) - π(v) = 0

Now let x $\in$ im (π)

x = π (y)

Then π(x) = $\pi(\pi$(y)) = π^2 (y) = π (y) = x.

$\Rightarrow$ π restricted to im(x) is identity.

Hence the proof.

MASCHKE'S THEOREM:

Let K be a field and G be a finite group.Suppose that the characteristics of K does not divide | G |. ie, | G | is invertible in the field K.Let (ρ , V) be a finite dimensional representation of G.Let W be a G-invariant subspace of V.Then $\exists$ W$'$ a G-invariant subspace such that V = W $\oplus$ W$'$.

Proof

Let ρ : G $\rightarrow$ GL(V) be a representation.Let W be invariant subspace of V.

Let W_0 be a complement ie, V = $W_0 \oplus$ W.

We want to find a complement which is G-invariant.

Let π be a projection corresponding to this decomposition.

ie, $\pi(w_0) = 0$ and π(w) = W, $\forall$ w $\in$ W.

Define an endomorphism $\pi^{'}$: V $\rightarrow$ V by averaging technique as follows

$\pi^{'} = \frac{1}{|G|} \sum \rho(\mathrm{t})^{-1}\ \pi\ \rho(\mathrm{t})$

We claim $\pi^{'}$ is a projection.

Now $\pi^{'}$(v) $\subset$ W since $\pi\ \rho$(t)(v) $\subset$ W and W is G-invariant.

$\pi^{'}$(w) = w $\forall$ w $\in$ W.

$\pi^{'}(\mathrm{w}) = \frac{1}{|G|} \sum \rho(t^{-1})\ \pi\ \rho(\mathrm{t})(\mathrm{w})$

$= \frac{1}{|G|} \sum \rho(t^{-1})\ \pi\ (\rho(\mathrm{t})(\mathrm{w}))$

$= \frac{1}{|G|} \sum \mathrm{w}$

= w. [since ρ(t)(w) $\in$ W and π takes it to itself].

Let v $\in$ V , then $\pi^{'}$ (v) $\in$ W

Hence $\pi^{'2}$ (v) = $\pi^{'}$ ($\pi^{'}$(v) = $\pi^{'}$(v).

as we have $\pi^{'}$(v) $\in$ W and $\pi^{'}$ takes any element of W to itself . Hence $\pi^{'} = \pi^{'}$.

Now we write decomposition of V with respect to $\pi^{'}$, say V = $W^{'} \oplus$ W where $W^{'}$ = ker ($\pi^{'}$) and W = im ($\pi^{'}$)

We claim that $W^{'}$ is G-invariant which will prove the theorem.For this assume that $\pi^{'}$ is a G-invariant homomorphism

ie , $\pi^{'}$(ρ (g)(v) = ρ (g)($\pi^{'}$(v)) $\forall$ g $\in$ G and v $\in$ V.

$\pi^{'}(\rho(\mathrm{g})(\mathrm{v})) = \frac{1}{|G|} \sum \rho(\mathrm{t})^{-1}\ \rho(\mathrm{t})\ (\rho(\mathrm{g})(\mathrm{v}))$

$= \frac{1}{|G|} \sum \rho(\mathrm{g})\ \rho(\mathrm{g})^{-1}\ \rho(\mathrm{t})^{-1}\ \pi\ \rho(\mathrm{t})\ (\rho(\mathrm{g})(\mathrm{v}))$

$= \rho(\text{g}) \ \frac{1}{|G|} \ \rho(\text{tg})^{-1} \ \pi \ \rho(\text{tg}) \ (\text{v})$

$= \rho(\text{g}) \ (\pi^{'} \ (\text{v}))$

Therefore $W^{'}$ is G-invariant.

Let w$^{'}$ ∈ W$^{'}$. We have to show that ρ(g)(w$^{'}$) ∈ W$^{'}$.

$\pi^{'} \ (\rho(\text{g})(w^{'})) = \rho \ (\text{g})(\pi^{'}(\text{w}^{'})) = \rho \ (\text{g})(0) = 0$

Therefore W$^{'}$ is an invariant compliment of W.

Hence the theorem.

'

PREPOSITION:

Let K be a field and G be a finite group with characteristics of K does not divide —G—. Then every finite dimensional representation of G is completely reducible.

PROOF:

Let $\rho :$ G → GL(V) be a representation.

We use induction on the dimension of V to prove this.

Let dim(V) = 1.

It is easy to verify that one-dimensional representation is always irreducible.Let V be of dimension n ≥ 2.

If V is reducible we have nothing to prove.So we assume that V has a G-invariant proper subspace , say W with 1 ≤ dim(V) ≤ n-1.

By Maschke's theorem we can write V = W ⊕ W$^{'}$, where $W^{'}$ is also G-invariant.But dim(W) and dim($W^{'}$) are less than n.

Therefore , by induction hypothesis they can be written as direct sum of irreducible representations.

DEFINITION 2.4:

G-MAP:

Let (ρ , V) and ($\rho^{'}$, V) be two representations of G over the field K.A linear map T:V $\rightarrow$ $V^{'}$ is called a G-map (between 2 representations) if it satisfies the following

$\rho^{'}$(t)T = T ρ(t) $\forall$ t $\in$ G.

The G-maps are also called interwiners.

SCHUR'S LEMMA:

Let (ρ , V) and ($\rho^{'}$, V) be two irreducible representations of G (of dimension ≥ 1). Let T : V $\rightarrow$ $V^{'}$ be a G-map.Then either T = 0 or T is an isomorphism. Moreover if it is non-zero , then T is an isomorphism if and only if the two representations are equivalent.

PROOF:

Let us consider the subspace ker(T).

We claim that it is a G-invariant subspace of V.

For this let v $\in$ ker(T).

Then T ρ(t)(v) = $\rho^{'}$ (t) T(v) = 0

$\Rightarrow$ ρ (t) (v) $\in$ ker(T) $\forall$ t $\in$ G

By applying Maschke's theorem on the irreducible representation V , we get either ker(T) = 0 or ker(T) = V.

In the case ker(T) = V , the map T = 0.

Therefore we may assume that ker(T) = 0. ie, T is injective.

Npw consider the subspace im(T) $\subset V'$.We claim that im(T) $\subset$ V$'$ is also G-invariant.

For this let y = T(x) $\in$ im(T).

Then ρ' (t) y = ρ' (t) T(x) = T ρ(t) T(x) $\in$ im(T) $\forall$ t $\in$ G.

Hence im(T) is G-invariant.

Again by applying Maschke's theorem for irreducible representation V$'$ we get either im(T) = 0 or im(T) = V$'$, which proves that T is an isomorphism.

<u>COROLLORY:</u>

Let (ρ , V) be an irreducible representation of G over C. Let T : V $\rightarrow$ V be a G-map.Then T = λId for some $\lambda \in$ C and Id is the identity map on V.

<u>PROOF:</u>

Let λ be an eigen value of T corresponding to the eigen vector of v $\in$ V
ie, T(v) λv

Consider the subspace W = ker(T - λ Id) . We claim that W is a G-invariant subspace.

Since T and scalar mutiplications are G-maps so is T-λ. Hence, the kernal is G-invariant (by schur's lemma).

Since, W $\neq$ 0 and is G-invariant , by Maschke's theorem W = V

$\Rightarrow$ T = λ Id.

Chapter 3

REPRESENTATION THEORY OF FINITE ABELIAN GROUP OVER C

PREPOSITION:

Let K = C and G be a finite abelian group. Let (ρ , V) be ab irreducible representation of G. Then dim(V) = 1.

PREPOSITION:

Let K = C and G be a finite abelian group. Let ρ : G $\rightarrow$ GL(V) be a representation of dimension n. Then we can choose a basis of V such that ρ (G) is contained in diagonal matrices.

PROOF:

Since V is a representation of finite group, we can write V as a direct sum of G-invariant irreducible ones by Maschke's theorem. ie, V = $W_1 \oplus W_2 \oplus .... \oplus W_r$.

Using Schur's lemma, we can conclude that dim(W_i) = 1 $\forall$ i and hence we get r = n. By choosing a vector in each W_i we get the required result.

COROLLORY:

Let G be a finite group. Let $\rho :$ G $\rightarrow$ GL(V) be a representation. Let g $\in$ G. Then there exists a basis of V such that the matrix of ρ(g) is diagonal.

PROOF:

Consider H = ¡g¿ $\subset$ G and $\rho :$ H $\rightarrow$ GL(V) the restriction map.Since H is abelian , we can simultaneously diaganalise elements of H.[Using the preposition " Let K = C and G be a finite abelian group. Let $\rho :$ G $\rightarrow$ GL(V) be a representation of dimension n. Then we can choose a basis of V such that ρ(V) is contained in diagonal matrices."] This proves the required result.

THEOREM:

Let G be a finite group.Every irreducible representation of G over C is one-dimensional iff G is an abelian group.

PROOF:

Let all irreducible representation of G over C be of dimension one. Consider the regular representation $\rho :$ G $\rightarrow$ GL(V) where V = $C_{[G]}$.

We have if —G— $\geq$ 2 this representation is reducible and is an injective map.

Using Maschke's theorem , we can write V as a direct sum of irreducible ones and they are given to be of dimension 1.

Therefore $\exists$ a basisi $V_1, V_2, ...V_n$ of V such that subspace generated by each basis vectors are invariant.Therefore ρ(G) consist of diagonal matrices with respect to this basis which is an abelian group.

Hence G $\cong$ ρ(G) is an abelian group.

DEFINITION 3.1:

SUBREPRESENTATIONS:

If we have a representation (ρ , V) and W is a G-invariant subspace , then we can define a subrepresentation (ρ , W) by ρ(t)(w) = ρ(t)(w)

DEFINITION 3.2:

ADJOINT REPRESENTATION:

Let ρ : G $\rightarrow$ GL(V) be a representation.We define the adjoint representation (ρ^* , V*) as follows: ρ^* : G $\rightarrow$ GL(V*) where ρ^* (g) = ρ (g^{-1})*

NOTE:
‘
Let V be a vector space over K with a basis $e_1, e_2, ..., e_n$. A linear map f : V $\rightarrow$ K is called a linear functional.We denote V* = (V,K) , the set of all linear functionals.

We define operations on V* : (f_1 + f_2) (v) = f$_1$(v) + f_2(v) and (λf) (v) = λf(V) and it becomes a vector space . The vector space V* is called dual space of V.

DEFINITION 3.3:

Let V and V' be two vector space over K.Tensor product of two vector spaces V and V' is a vector space V $\otimes$ V' = { $\sum_{i=1}^{r} V_i \otimes V_i'$ / $V_i \in$ V , $V_i' \in V'$ } with the following properties

1. $\sum_{i=1}^{r} (V_i \otimes V_i') + \sum_{i=1}^{s} (W_i \otimes W_i') = V_1 \otimes V_1' + ... + V_r \otimes V_r' + W_1 \otimes W_1' +....+ W_s \otimes W_s'$.

2. $(V_1 + V_2) \otimes V'$ = V $\otimes$ V_1' + V $\otimes$ V_2'

V $\otimes$ $(V_1' + V_2')$ = V $\otimes$ V_1' + V $\otimes$ V_2'

3. $\lambda \sum_{i=1}^{r} (V_i \otimes V_i') = \sum_{i=1}^{r} \lambda V_i \otimes V_i' = \sum_{i=1}^{r} V_i \otimes \lambda V_i'$

NOTE:

If (ρ , V) is a representation of G , then $V\otimes^n = V\otimes$ V$\otimes$......$\otimes$V, sym^{n}V, λ^n(V) are also representations of G. ie, Starting from one representation we get many representations.If we start from an irreducible representation the above constructed representations need not be irreducible,but they often contain other irreducible representations.

Direct sum decomposition of tensor representation is an important topic of study. It happens that we need much smaller number of representations (called fundamental representations) of which tensor product contains all irreducible representations

Let (ρ , V) be a representation of the group G. Let H be a subgroup. Then (ρ , V) is a representation of H denoted as (ρ_H , V).

Let N be a normal subgroup of G.Then any representation of G—N gives rise to a representation of G. Moreover , if the representation of G—N is irreducible , then the representation of G remains irreducible.(Restriction of representations).

Chapter 4

CHARACTER THEORY

DEFINITION 4.1:

Let C_G , space of all complex valued functions on G which is a vector space of dimension —G—. We define the inner product as

$<,> : C_G \times C_G \to$ C by $<f_1$, $f_2> = \frac{1}{|G|} \sum f_1(t)\ f_2(t)$

DEFINITION 4.2:

Let (ρ, V) be a representation of G.The character of (corresponding to) ρ is a map χ : G → C defined by χ(t) = tr(ρ(t)) , where tr is the trace of corresponding matrix

DEFINITION 4.3:

A function f : G → C is called a class function if f is constant on the conjugacy classes of G. We denote the set of class functions on G by $'$H.

DEFINITION 4.4:

Let G be a finite group.Let W_1, W_2......W_h be irreducible representations of G of dimension n_1,n_2.....n_h over C. Let χ_1 , χ_2,.....χ_h are corresponding characters called irreducible characters of G.

THEOREM:

Let χ be a character of representation (ρ , V) . Then $<\chi , \chi>$ is a positive integer and $<\chi \ \chi> = 1$ iff V is irreducible.

DEFINITION 4.5:

Let G be a finite group and χ_1 , χ_2,......χ_h be the irreducible characters of dimension n_1 , n_2......n_h respectively.Let L be the left regular representation of G with corresponding character l.

LEMMA:

Let f $\in {}'$H be a class function on G. Let (ρ , V) be an irreducible representation of G of degree n with character χ. Let us define $\rho_f = \sum$ f(t) ρ(t) .

Then $\rho_f = \lambda$ Id where $\lambda = \frac{|G|}{n} <$ f , X $>$

PROOF:

We claim that ρ_f is a G-map and Schur's lemma to prove this.

For any g $\in$ G , we have

$\rho(g)\ \rho_f\ \rho(g^{-1}) = \sum$ f(t) ρ(g) ρ(t) $\rho(g^{-1})$

$= \sum$ f(t) ρ(gtg^{-1})

$= \sum$ f(g^{-1}sg) ρ(s)

$= \rho_f$

Hence ρ_f is a G-map.

Therefore from Schur's lemma , we get $\rho_f = \lambda$ Id for some $\lambda \in$ C.

Taking trace on both sides,

λ.n = tr(ρ_f)

= $\sum$ f(t) tr(ρ(t))

= $\sum$ f(t) χ(t)

= | G | $\frac{1}{|G|}$ f(t) $\chi(t^{-1})$

= | G | $<$ f , χ $>$

$\Rightarrow \lambda = \frac{|G|}{n} <$ f , $\chi >$

APPLICATIONS

Ramanujan made many contributions to the theory of partitions.Hardy and Ramanujan developed the circle method and published asymptotic formula like p(n).The actual values of p(n) , for small n , are much smaller than the asymptotic values.Representation theory especially of symmetric group plays a role to provide easy but close lower bounds for p(n).

The Dedikind zeta function of an algebraic number field is an invariant which plays an important role in density theorems for ramfication of primes like Frobenius density theorem and the Chebychev's density theorem.A simple result from representation theory of finite groups provides a method to construct non-isomorphic number fields with the same zeta function.This also provides a footing to discuss and prove special cases of Dedikind's conjucture.

REFERENCES

1.Anupam Singh , "Representation theory of finite groups", Indian institute of science education and research , Pune.

2. William Fulton and Joe Harris , " Representation theory, A first course" , Springer International edition.

3. James E Humphreys , "Introduction to Lie algebras and representation theory" , Springer international edition.

4. https://en.wikipedia.org/wiki/Representation theory of finite groups.

5. https://en.wikipedia.org/wiki/Character theory.