PERFECT GRAPH

A Dissertation submitted in partial fulfillment of the

requirement for the award of

DEGREE OF MASTER OF SCIENCE

IN MATHEMATICS

By

NAISEENA T.A

REGISTER NO: SM16MAT011

(2016 – 2018)

DEPARTMENT OF MATHEMATICS

ST.TERESA'S COLLEGE, (AUTONOMOUS)

ERNAKULAM, KOCHI - 682011

APRIL 2018

DEPARTMENT OF MATHEMATICS

ST.TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM

CERTIFICATE

This is to certify that the dissertation entitled "PERFECT GRAPH" is a bonafide record of the work done by NAISEENA T.A under my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

Mrs. Nisha Oommen (Supervisor)

Assistant Professor

Department Of Mathematics

St.Teresa's College,(Autonomous)

Ernakulam

Smt Teresa Felitia (HOD)

Associate Professor

Department of Mathematics

St. Teresa's College (Autonomous)

Ernakulam

External Examiners:

1. ....

2. ....

DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Mrs. Nisha Oommen, Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

ERNAKULAM | NAISEENA T.A

APRIL 2018 | **REGISTER NO: SM16MAT011**

ACKNOWLEDGEMENT

I express my sincere gratitude to my guide Mrs. Nisha Oommen, Assistant professor, Department of Mathematics, St. Teresa's college (Autonomous),Ernakulam for the successful completion of this work.

I further express my sincere gratitude to Smt Teresa Felitia, Head of the department of Mathematics, St Teresa's College (Autonomous), Ernakulam and other teachers of the department for their encouragement, suggestions and assistance in taking up this dissertation.

I finally thank my parents ,friends and all my wellwishers who had supported me during the project.

ERNAKULAM

APRIL 2018

NAISEENA T.A

REGISTER NO: SM16MAT011

CONTENTS

INTRODUCTION

The theory of perfect graphs relates the concept of graph colourings to the concept of cliques. A graph is perfect if the chromatic number and the clique number have the same value for each of its induced subgraphs. Perfect graphs include many important graphs classes including bipartite graphs, chordal graphs, and comparability graphs.

The notion of perfect graphs are introduced by Claude Berge in 1960. He also conjectured that a graph is perfect iff it contains, an an induced subgraph, neither an odd cycle of length atleast five nor its complement. This conjecture become known as the strong perfect graph conjecture and attempts to prove it contributed much to the development of graph theory in the past forty years. Chudnovsky, Robertson, Seymour and Thomas were recently able to prove the strong perfect graph conjecture in its full generality.

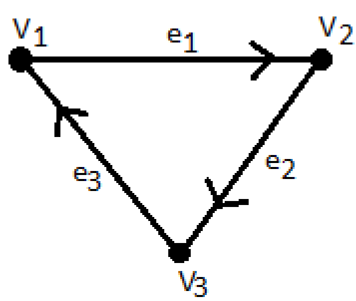
After remaining unsolved for more than forty years, it can now be called Strong perfect graph theorem.

CHAPTER 1

PRELIMINARIES

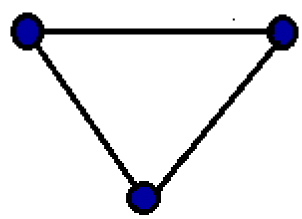
Graph

A graph is an ordered triple (V(G),E(G),Ψ(G)) consisting of a non empty set V(G) of vertices.Set E(G) of edges disjoint from V(G) and an incidence fuction Ψ(G) that associates with each edge of G and unordered pair of vertices of G.



Undirected graph

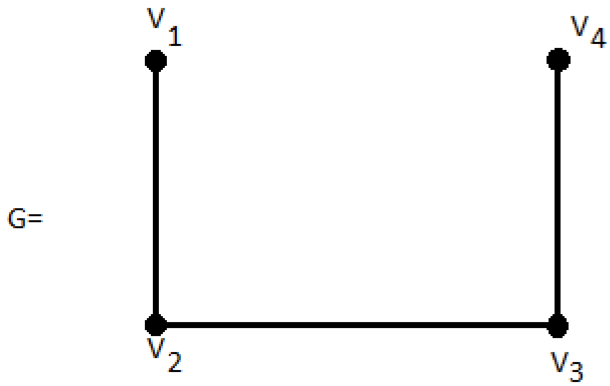
An undirected graph is a graph in which edges have no orientation. The edge (x,y) is identical to the edge (y,x). The maximum number of edges in an undirected graph without a loop is $\frac{n(n-1)}{2}$.

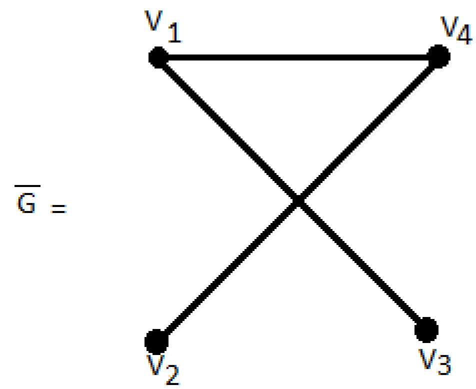


Undirected graph

Complement graph

The complement of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent iff they are not adjacent in G.
i.e; to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there.





Induced subgraph

Let G= (V,E) be any graph and let S $\subset$ V be any subset of vertices of G. Then the induced subgraph G[S] is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both end points in S.

Colouring

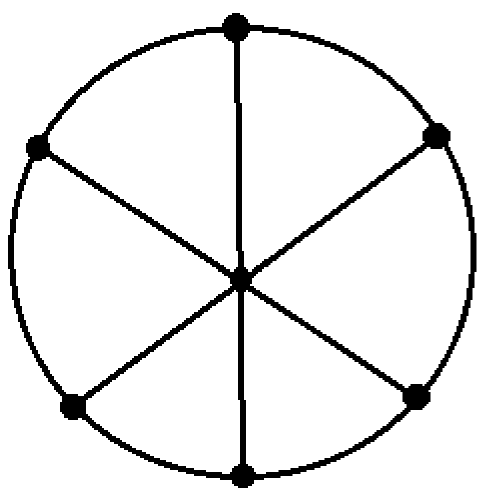
Graph colouring is a special case of graph labelling. It is a way of colouring the vertices of a graph such that no two adjacent vertices share the same colour; this is called a vertex colouring. Similarly, an edge colouring assigns a colour to each edge so that no two adjacent edges share the same colour.

Independent set

An independent set is a set of vertices in a graph, no two of which are adjacent. i.e; It is a set S of vertices such that for every two vertices in S, there is no edge connecting the two.

Chromatic number χ(G)

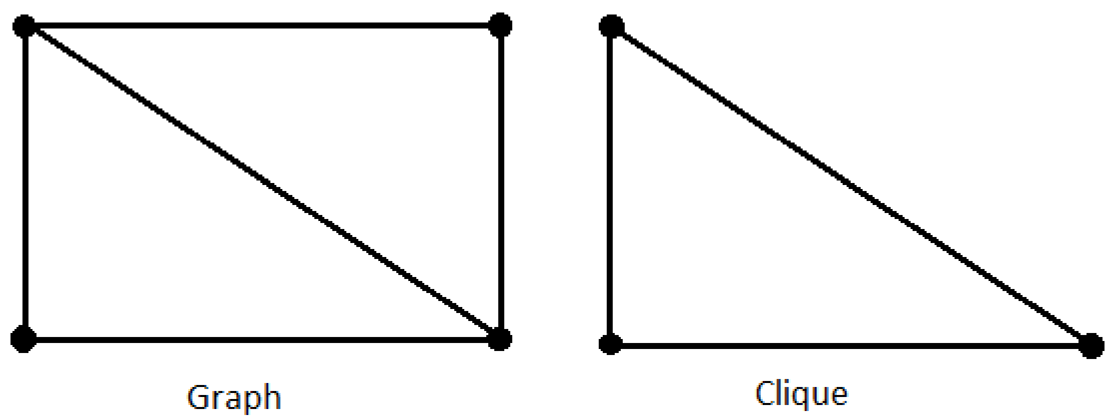
The chromatic number of G is the minimum K for which G is K colourable.



3 Chromatic

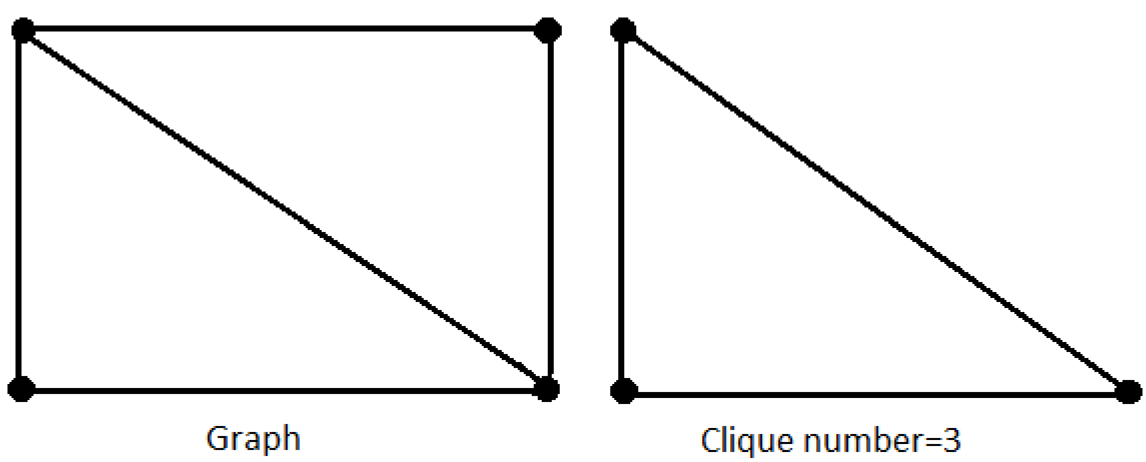
Clique

For any graph G, a complete subgraph of G is called a clique of G.



Clique number $\omega(\mathrm{G})$

Clique number of a subgraph of G is the size of the largest clique contained in G.

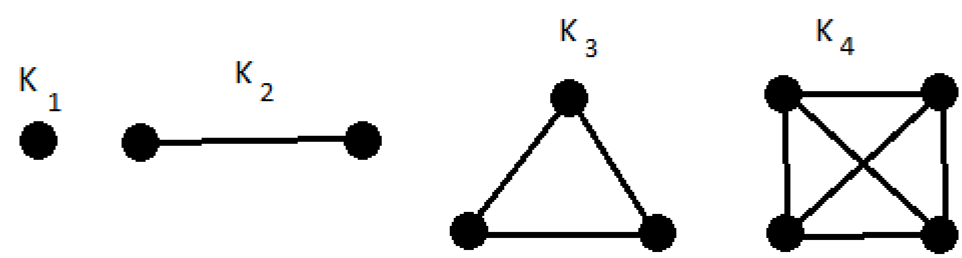


Clique cover

In graph theory, a clique cover or partition into cliques of a given undirected graph is a partition of the vertices of the graph in to cliques, subsets of vertices within which every two vertices are adjacent. A minimum clique cover is a clique cover that uses as few cliques as possible.

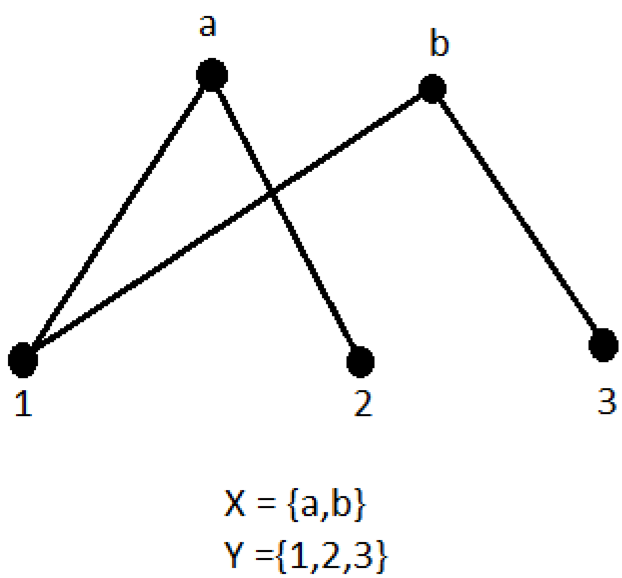
Complete graph

A simple graph G is said to be complete if its each pair of distinct vertices is join by an edge. A complete graph with n vertices is usually denoted by K_n and has exactly $\frac{n(n-1)}{2}$ edges.



Bipartite graph

A graph is bipartite if its vertex set can be partitioned into two non empty subset X and Y such that each edge has one end in X and other end in Y. Such a partition (X,Y) is called a bipartition of the graph.



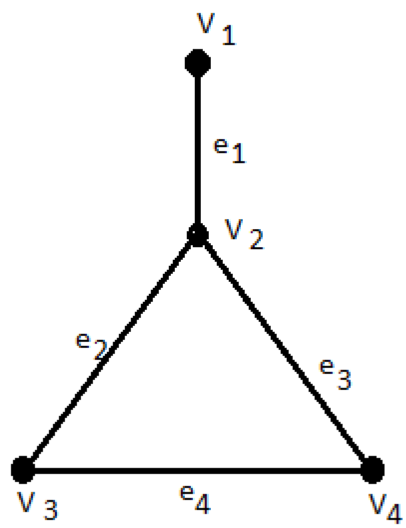
Line graph

If G is a graph, we can form the line graph L(G)= H from G as follows

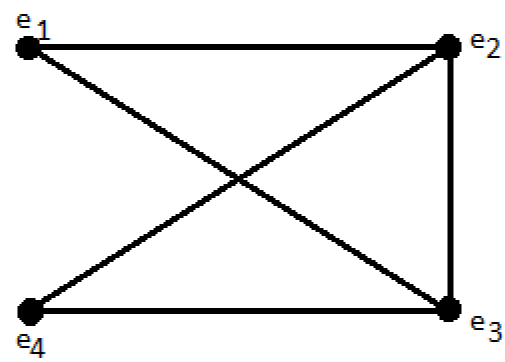
• The vertex set of H is the edge set E(G) of G.

• There is an edge between two elements of V(H) iff the two corresponding edges in E(G) share a common vertex.

G=



L(G)=



Partially ordered set

A partially ordered set (poset) consists of a set together with a binary relation indicating that, for certain pairs of elements in the set, one of the elements precedes the other in the ordering. To be partial order, a binary relation must be reflexive, antisymmetric, and transitive.

Chain

A chain in S is a subset C of S in which each pair of elements is complete. i.e; C is totally ordered.

Antichain

An antichain is a subset of a partially ordered set such that any two distinct elements in the subset are incomparable.

Odd hole

Cycle of odd length atleast 5 is called an hole.

Odd Antihole

An induced subgraph that is the complement of an odd hole is called an odd antihole.

CHAPTER 2

PERFECT GRAPH

Definition 2.1
A graph is perfect if χ(H)=ω(H) for every induced subgraph H of G.
i.e; In graph theory, a perfect graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Note
if H is an induced subgraph of G, every induced subgraph of H is also an induced subgraph of G.Thus if G is a perfect graph, then every induced subgraph of G is also perfect. The perfection is preserved under graph isomorphism.
If G and G' are isomorphic, then G is perfect iff G' is perfect.

Examples

• The most trivial class of graphs that are perfect are the edgeless graphs.

i.e; The graph with V=1,2,...n and E=Φ these graphs and all of their subgraphs have both chromatic number and clique number 1.

• Any bipartite graph is perfect. This is trivial as

i) any induced subgraph of a bipartite graph is bipartite

ii) the largest clique in a bipartite graph is 2(or 1 if the graph is empty) while the number of colors needed is 2(or 1 if the graph is empty)

Theorem 2.2

All complete graphs are perfect.

proof

We will prove this theorem using induction on the order of the graph. The complete graph of order 1 (K_1) is perfect.

Since $\omega(K_1)= \chi(K_1)= 1$ and K_1 has no induced subgraphs.

Assume that complete graphs of order n are perfect. Let K_{n+1} be a complete graph of order n+1 and let H be an induced subgraph of K_{n+1}. All induced subgraphs of complete graphs are complete graphs. If H is a proper

subgraph of K$_{n+1}$, it is a complete graph of order n or less and is perfect by the induction hypothesis. Thus ω(H)= χ(H).
Finally , if H= K$_{n+1}$, ω(K$_{n+1}$)= χ(K$_{n+1}$)= n+1
Thus all complete graphs are perfect.

Note
Every order 1 graph is a complete graph,so every order1 graph is perfect.

Theorem 2.3
If G is biprtite graph, then the complement graph $\overline{G}$ of G is perfect.
Proof
Let G=((V$_1$,V$_2$),E) be a bipartite graph on n vertices, $\overline{G}$ be its complement,and $\overline{H}$ be any induced subgraph of $\overline{G}$ on the vertex set $\{V_1^1, ...V_k^1\} \cup \{V_1^2, ...V_m^2\}$ Then, $\overline{H}$ is the graph with edges between all of the V_i^{1}'s in it,as well as edges between all of the V_j^{2}'s in it,and also an edges between a V_i^1 and a V_j^2 iff such an edge did not exist in G.
Consequently,by definition,$\overline{H}$ is itself the complement graph of the induced subgraph H of G given by the same

vertex set $\{V_1^1,...V_k^1\} \cup \{V_1^2,...V_m^2\}$,which is a bipartite graph.
$\therefore$ any subgraph of $\overline{H}$ is itself complement graph of a bipartite graph. Thus,if we can prove that $\chi(\overline{H}) = \omega(\overline{H})$ for any complement graph to a bipartite graph,then we will have proven that this in fact hold for all of G‘s subgraphs.(as they are all also complement graphs to bipartite graphs) and thus shown that $\overline{G}$ is perfect.

Proposition 2.4
If G is a bipartite graph,then L(G) is perfect.

Theorem 2.5
All open chains are perfect.
Proof
Let G=(V,E) be an open chain.Let H be an induced subgraph of G. There are two potential cases: Case 1: H contains no neighboring vertices,For example Fig 2.4. If H contains no neighboring vertices,ω(H)=1. The function C(V$_i$)= red for all V$\in$H is a 1-colouring of H. So, ω(H)=χ(H).
Case 2: H contains neighboring vertices,For example Fig 2.5. There are no cliques of size 3 or greater,so $\omega(H)$=2.

The function C (defined below) is a 2-colouring of H. So, χ(H)=ω(H).

$$c(v)=\begin{cases} \text{red, i is odd} \\ \text{blue, i is even} \end{cases}$$

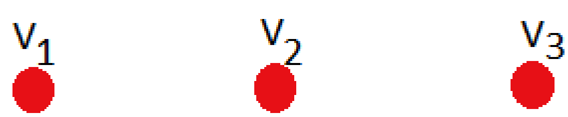


Fig 2.4 An induced subgraph with no neighbours

Fig 2.5 An induced subgraph with neighbours

CHAPTER 3

FAMILIES OF GRAPHS THAT ARE PERFECT

Triangulated (Chordal) graph

A graph G in which each cycle of length atleast 4 has a chord is called triangulated or chordal graph.

Proposition 3.1

A graph is triangulated iff every minimal vertex separator induces a complete subgraph.

Lemma 3.2

Let S be a vertex cut of a connected graph G and for 1< i <r, let C_i (on vertex set V_i) be the components of G-S.Let G_i= $< V_i \cup S_i >$ be the S component of G. Then if $< S >$ is a clique,$\chi(G) = \overset{max}{i} \ \chi(G_i)$ and

$\omega(G) = \overset{max}{i} \ \omega(G_i)$

Corollary 3.3

If G_i is perfect then G is perfect.

Proposition 3.4

Every triangulated graph is perfect.

Proof

By proposition 3.1, every triangulated graph G has a vertex cut S inducing a complete subgraph. Using induction on the number of vertices, the induction hypothesis gives that every S- component is perfect. Then by corollary 3.3 , G is perfect.

Definition

i) A graph G is strongly perfect if every induced subgraph H of G has an independent set meeting all cliques in H.
ii) An independent set is good if it meets all cliques of the graph. A graph G is very strongly perfect if every vertex of each induced subgraph H of G belongs to a good independent set oh H.
iii) A graph is co-strongly (very strongly) perfect iff G and $\overline{G}$ are strongly (very strongly) perfect.

Proposition 3.5

Every strongly perfect graph is perfect.

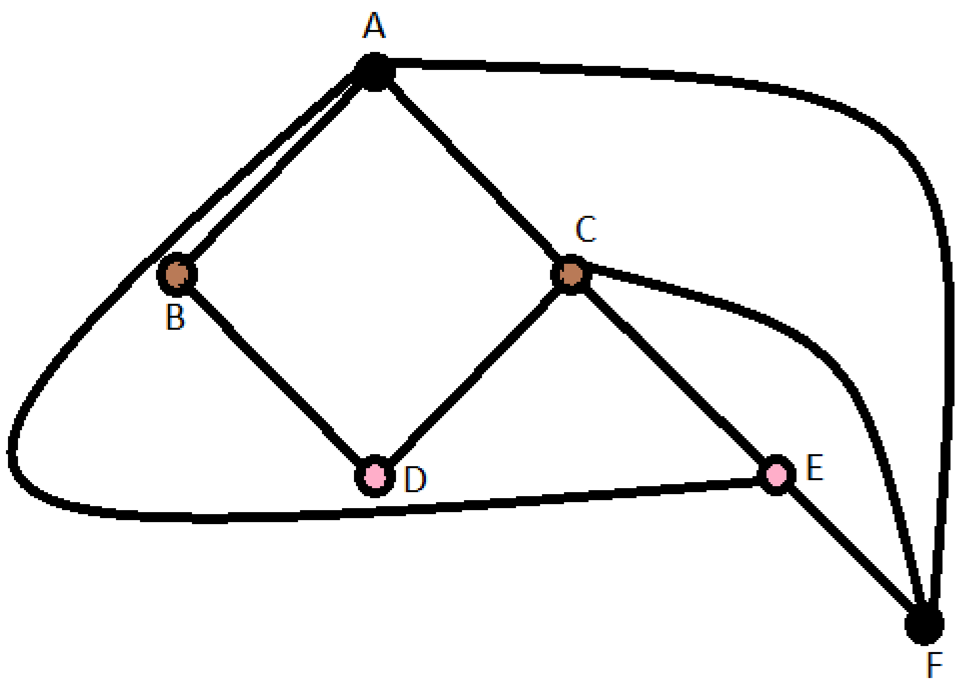
Proof

Let G be a strongly perfect and H be an induced subgraph

of G. Then H has a good independent set S_1 meeting all cliques of H, so that $\omega(H - S_1) = \omega(H) - 1$
Let $H_1 = H - S_1$ and $\omega(H) = \omega$
Then H_1 is a strongly perfect graph and has good independent set S_2.Thus $\omega(H_1 - S_2) = \omega(H_1) - 1 = \omega - 2$ and$H_2 = H_1 - S_2$ is a strongly perfect graph.
Proceeding this way, we get ω independent sets $S_1, S_2 ... S_\omega$ covering all the vertices of H, so that $\chi(H) \leq \omega(H)$.
But $\chi(H)$ is also greater than or equal to $\omega(H)$, giving $\chi(H) = \omega(H)$ so that G is χ - perfect and hence perfect.

Definition (Comparability graphs)
Let P be a set with partial order $\leq$ and let G = (P,E), G is comparable if E = $\{ \{x,y\} : x,y \in P , x \perp y, x \neq y\}$
It is useful to note that all cliques of a comparability graph are chains , since if K is a clique and x,y $\in$K, then x and y are neighbours so x $\perp$y. Similarly independence sets of comparability graphs are antichains.

Proposition 3.6
Every comparability graph is perfect.
Proof
We show that a comparability graph G = (V,E) is χ- perfect. Let H = (V,E) be a transitive orientation of the edges of G. For each v $\in$ V, let t(v) = 1+ length of a longest



directed path from V to H.
If max{ t(v)/ v∈ V} is k, then G has a k-clique (induced by the vertices of the longest path from v,a vertex for which t(v) =k).
Also G does not have a (k+1) clique, since in that case it will have a path of length k passing through the vertices of the clique.
Thus ω(G)=k.
Colour each of vertex v of G with colour t(v). This is a proper colouring using k colours, since if u and v are ad-

jacent in G and uv is directed from u to v in H, then t(u) > t(v). Thus χ(G) $\leq$ k= ω(G).
Since χ(G) $\geq$ ω(G),we have χ(G)= ω(G).
The same argument can be applied to every induced subgraph of G. Hence G is χ- perfect.

Definition (Incomparability graphs)
Let P be a set with partial order $\leq$ and let G= (P,E). G is incomparable graph if E= $\{ \{x,y\}: x,y \in P, x \| y \}$
It is useful to note that all cliques of an incomparable graph are antichains, since, if k is a clique and x,y$\in$k, then x and y are neighbours so x$\|$y. Similarly, independence sets of incomparable graphs are chains.

Proposition 3.7
Incomparability graphs are perfect.
Proof
Proving incomparability graphs are perfect by showing ω(H)= χ(H) is difficult, since there is no analogue to a longest increasing chain to use as the basis of colouring. Instead, we note that the converse of any incomparability graph is a comparability graph. Since all the comparability graphs are perfect, by the weak graph theorem, all incomparability graphs are perfect.

Definition (Interval graphs)

A graph is an interval graph if it is the intersection of a set of intervals of any total order. It is a proper interval graph if no interval is contained in another. It is a nested interval graph if for intervals I_j, I_k, $I_j \cap I_k \neq \phi \Rightarrow I_j \subset I_k$ or $I_k \subset I_j$.

Proposition 3.8

Every interval graph is triangulated.

Proof

Let G be an interval graph and if possible let c= $v_1, v_2, v_3, \ldots v_k, v_1$ be a cycle of length k≥4 without chord. Let I_i be the intervals represented by V_i, $1 \leq i \leq k$.

By assumption about cycle $I_{i-1} \cap I_{i+1} = \phi$ and $I_{i-1} \cap I_{i+1} \neq \phi$, for $2 \leq i \leq k$.

For cycle to be complete, $I_i \cap I_k$ should be non empty but this would imply $I_j \cap I_k \neq \phi$, for $2 \leq j \leq k$, contradicting that G has no chords. Thus G is triangulated.

Proposition 3.9

The complement of every interval graph is a comparability graph.

Proof

Let $\{I_v \ / \ v \in V\}$ be an interval representation of the interval graph G=(V,E). Define orientation F for $\overline{G}$=(V,$\overline{E}$) as

follows.
uv$\in$F $\Leftrightarrow$ I_V for uv$\in$ $\overline{E}$.
Since uv $\in \overline{E} \Rightarrow I_u \cap I_v = \phi$, this is certainly an orientation of every edge of $\overline{G}$. Also uv$\in$ F and uw$\in$F $\Rightarrow$ ($I_u < I_v$ and $I_v < I_w$) $\Rightarrow$ $I_u < I_w$ $\Rightarrow$ uw $\in$ F so that F is a transitive orientation. Thus $\overline{G}$ is a comparability graph.

Proposition 3.10
Every interval graph is perfect.
Proof
Since every interval graph is triangulated and every triangulated graphs are perfect. We can say that every interval graph is perfect.

CHAPTER 4

PERFECT GRAPH THEOREM

In graph theory, the perfect graph theorem states that undirected graph is perfect iff its complement graph is also perfect. This result had been conjectured by Berge (1961,1963), and it is sometimes called the weak perfect graph theorem to distinguish it from the strong perfect graph theorem.

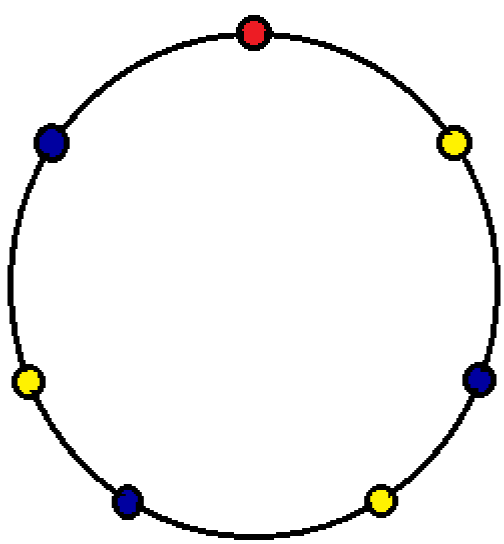
Theorem statement

A perfect graph is an undirected graph with the property that, in every one of its induced subgraphs, the size of the largest clique equals the minimum number of colours in a colouring of the subgraph. Perfect graphs include many important graphs classes including bipartite graphs,chordal graphs and comparability graphs.

The complement of a graph has an edge between two vertices iff the original graph does not have an edge between the same two vertices. Thus, a clique in the original graph becomes an independendent set in the complement and a colouring of the original graph becomes a clique cover of

the complement.
The perfect graph theorem states:
The complement of a perfect graph is perfect.
Equivalently, in a perfect graph, the size of the maximum independent set equals the minimum number of cliques in a clique cover.

Example



Let G b a cycle graph of odd length greater than 3. Then G requires atleast 3 colours in any colouring, but has no triangle, so it is not perfect. By the perfect graph theorem, the complement of G must therefore also not be perfect.

Applications

In a non trivial bipartite graph, the optimal number of colours is 2, and the maximum clique size is also 2. Also, any induced subgraph of a bipartite graph remains bipartite. Therefore bipartite graphs are perfect.
In n-vertex bipartite graphs,a minimum clique cover takes the form of a maximum matching together with an additional clique for every unmatched vertex, with size n-M, where M is the cardinality of the matching.
Thus, in this case, the perfect graph theorem implies Konig's theorem(describes an equivalence between the maximum matching problem and the minimum vertex cover problem in bipartite graphs) that the size of a maximum independent set is also n-M, a result that was a major inspiration for Berge's formulation of the theory of perfect graphs.
Mirsky's theorem characterizing the height of a partially ordered set in terms of partitions into antichains can be formulated as the perfection of the comparability graph of the partially ordered set, and Dilworth's theorem characterizing the width of a partially ordered set in terms of partitions into chains can be formulated as the perfection of the complements of these graphs. Thus, the perfect

graph theorem can be used to prove Dilworth's theorem from the proof of Mirsky's theorem, or vice versa.

Strong perfect graph theorem

In graph theory, the strong perfect graph theorem is a forbidden graph characterization of the perfect graphs as being exactly the graphs that have neither odd holes nor odd antiholes. It was conjectured by Claude Berge in 1961. A proof by Maria Chudnovisky, Neil Robertson, Paul Seymour, and Robin Thomas was announced in 2002 and published by them in 2006.

Theorem statement

A perfect graph is a graph in which, for every induced subgraph, the size of the maximum clique equals the minimum number of colours in a colouring of the graph. In his 1961 and 1963 works,Claude Berge observed that it is impossible for a perfect graph to contain an odd hole, an induced subgraph in the form of an odd- length cycle graph of length 5 or more, because odd holes have clique number 2 and chromatic number 3. Similarly, he observed that perfect graphs cannot contain odd antiholes, induced subgraphs complementary to odd holes: an odd antihole with 2k+1 vertices has clique number k and chromatic number k+1, again impossible for a perfect graphs. The graphs having neither odd holes nor odd antiholes became known as the Berge graphs.

Berge conjectured that every Berge graph is perfect, or equivalently that the perfect graphs and the Berge graphs define the same class of graphs. This became known as the strong perfect graph conjecture, until its proof in 2002, when it was renamed the strong perfect graph theorem.

CONCLUSION

Perfect graphs are one of the deepest and most fascinating graph theory topic to emerge in the last 20^{th} century. In this paper, we introduced the concept of perfect graphs. We proved that the complete graphs, open chains, chordal graphs, comparability graphs, incomparability graphs, interval graphs are all perfect. We also see the weak and strong perfect graphs and their applications. We have only touched the surface of what is possible with perfect graphs. There are several additional classes of perfect graphs that can be used in countless applications; research on perfect graphs and their applications is on going.

REFERENCES

[**1**] Douglas B West, *Introduction to Graph Theory*, *second edition*,,Published by Parson Education Inc, Published as Peason Prentice Hall.

[**2**] K.R Parthasarathy, *Basic Graph Theory*,Tata Mc Graw-Hill publishing company limited,New Delhi.

[**3**]Maria Chudnovsky, Neil Robertson,Paul Seymour,and Robin Thomas, *The strong perfect graph theorem.*

[**4**] Wikipedia.