A STUDY ON PELL'S EQUATION

A Dissertation submitted in partial fulfillment of the

requirement for the award of

DEGREE OF MASTER OF SCIENCE

IN MATHEMATICS

By

JOSMY THOMAS

REGISTER NO: SM16MAT006

(2016 – 2018)

DEPARTMENT OF MATHEMATICS

ST.TERESA'S COLLEGE, (AUTONOMOUS)

ERNAKULAM, KOCHI - 682011

APRIL 2018

DEPARTMENT OF MATHEMATICS

ST.TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM

CERTIFICATE

This is to certify that the dissertation entitled"A STUDY ON PELL'S EQUATIONS" is a bonafide record of the work done by JOSMY THOMASunder my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

SmtNishaOommen (Supervisor)   **Smt Teresa Felitia   (HOD)**

Assistant Professor   Associate Professor

Department Of Mathematics   Department of Mathematics

St.Teresa's College,(Autonomous)   St. Teresa's College (Autonomous)

Ernakulam   Ernakulam

External Examiners:

1. ....   2. ....

DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Smt.NISHA OOMMEN , Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

ERNAKULAM

JOSMY THOMAS

APRIL 2018

REGISTER NO: SM16MAT006

ACKNOWLEDGEMENT

I bow my head before God Almighty who showered his abundant grace on me to make this project success.

With great pleasure I express my sincere gratitude to my guide Smt.NishaOommen , Associate Professor , Department of Mathematics , St. Teresa's college , Ernakulam , for the successful completion of this work.

I further express my sincere gratitude to Dr.Sajimol Augustine M , Principal ,Smt. Teresa Felitia P.A, Head of the Department of Mathematics , St. Teresa's college , Ernakulam , and other teachers of the Department for their encouragement , suggestions , and assistance in taking up this dissertation.

I am most graceful to the Library staff for the help extended in accomplishing this work.

I finally thank my parents , friends , and all my well wishers who had supported me during the project.

ERNAKULAM   JOSMY THOMAS

APRIL 2018   **REGISTER NO: SM16MAT006**

CONTENTS

INTRODUCTION:

Number theory is that branch of mathematics that is concerned with the properties of numbers. For this reason, number theory, which has a 4000 years of rich history, has traditionally been considered as pure mathematics. The theory of numbers has always occupied a unique position in the world of mathematics. This is due to unquestionable historical importance of the subject. It is one of the few disciplines having demonstrable results that predate the very idea of a university or an academy. The natural numbers have been known to us for so long that mathematician Leopold Kronecker once remarked, " God created the natural numbers, and all the rest is the work of man". Far from being a gift from Heaven, number theory has a long and sometimes painful evolution.

The theory of continued fractions begins with Rafael Bombelli, the last of great algebraists of Renaissance Italy. In his L'Algebra Opera (1572) , Bombelli attempted to find square roots by using infinite continued fractions. One of the main uses of continued fraction is to find the approximate values of irrational numbers.

Srinivas Ramanujan has no rival in the history of mathematics. His contribution to number theory is quite significant. G.H.Hardy, commenting on Ramanujan's work, said "On this side (of Mathematics) most recently I have never met his equal, and I can only compare him with Euler or Jacobi".

Pell's equation $x^2 - dy^2 = 1$,was probably first studied in the case $x^2 - 2y^2 = 1$ .Early mathematicians, upon discovering that $\sqrt{2}$ is irrational, realized that although one cannot solve the equation $x^2 - 2y^2 = 0$ in integers,one can at least solve the "next best things". The early investigators of Pell equation were the Indian mathematicians Brahmagupta and Bhaskara. In particular Bhaskara studied Pell's equation for the values $d = 8, 11, 32, 61$ and 67 and Bhaskara found the solution $x = 1776319049, y = 2261590$, for d = 61 .

Fermat was also interested in the Pell's equation and worked out some of the basic theories regarding Pell's equation. It was Lagrange who

discovered the complete theory of the equation,$x^2 - dy^2 = 1$ .Euler mistakenly named the equation to John Pell. He did so apparently because Pell was instrumental in writing a book containing these equations. Brahmagupta has left us with this intriguing challenge: "A person who can, within a year, solve ,$x^2 - 92y^2 = 1$ is a mathematician."

In general Pell's equation is a Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a positive non square integer and has a long fascinating history and its applications are wide and Pell's equation always has the trivial solution $(x, y) = (1, 0)$, and has infinite solutions and many problems can be solved using Pell's equation.

Chapter 1

PRELIMINARIES:-

Definition 1.1 *Divisibility :*

If a and b are two integers, then we say that a divides b and we write $a \mid b$ if $b = aK$ for some integer K.

Definition 1.2 *Division Algorithm:*

Given two integers a and b with $b > 0$, there exists unique integers q and r satisfying

$$a = qb + r \ , 0 \leq r < b$$

The integers q and r are called quotient and remainder respectively in the division of a by b.

Definition 1.3 *Greatest Common Divisor:*

The greatest common divisor (m, n) of integers m and n is the largest integer which divides both m and n.

- *Given integers a and b, not both of which are zero, then there exist integers x and y such that*

$$d = (a, b) = ax + by$$

- *Euclid lemma: If $(a, b) = 1$ and $a \mid bc$, then $a \mid c$*
- *m and n are relatively prime if $(m, n) = 1$*

Euclidean Algorithm:

The Euclidean algorithm used to find the gcd of two integers, is the repeated application of division algorithm, starting with the numbers a and b , and terminating when a remainder of 0 occurs.

In general the Euclidean algorithm runs as follows:

a=bq$_1$+r$_1$,$0 \leq r_1 < b$

b=r$_1$q$_2$+r$_2$,$0 \leq r_2 < r_1$

r$_1$=r$_2$q$_3$+r$_3$,$0 \leq r_3 < r_2$

.
.
.

r$_i$=r$_{i+1}$q$_{i+2}$+r$_{i+2}$,$0 \leq r_{i+2} < r_{i+1}$

.
.
.

r$_{n-2}$=r$_{n-1}$q$_n$+r$_n$,$0 \leq r_n < r_{n-1}$

r$_{n-1}$=r$_n$q$_{n+1}$+0

Hence $r_n = (a, b)$

EXAMPLE:
To calculate $d = (1492, 1066)$ we have

1492=1×1066+426
1066=2×426+214
426=1×214+212
212=106×2+0

The last non zero remainder is 2. So $d = 2$

Linear Diophantine Equation:

A linear equation which is to be solved for integers is called a Diophantine equation. The linear Diophantine equation of the form $ax + by = c$ has solution if and only if $(a, b) \mid c$. Let $a, b \in Z$. Consider the linear Diophantine equation $ax + by = c$.

a) If $(a, b) \nmid c$, there are no solutions.

b) If $(a, b) \mid c$, there are infinitely many solutions of the form

$x = \frac{b}{d}k + x_0$

$y = \frac{a}{d}k + y_0$

where (x_0, y_0) is a particular solution and $k \in Z$

EXAMPLE:
6x+9y=21
since (6,9)=3 , 3|2 there are infinitely many solutions. By trial and error we find that, x=-7, y=7 is a particular solution.

Hence the general solution is given by

x=3k-7 , y=-2k+7 ; k $\in Z$

Prime number:

An integer p is called a prime number if its only positive divisors are 1 and p.

● An integer greater than 1 that is not a prime is termed as composite .

Fundamental Theorem of Arithmetic:

Every positive integer $n > 1$ can be expressed as a product of prime, this representation is unique, apart from the order in which the factors occur, that is $n = p_1 p_2 ... p_n$.

⋆ There exists infinitely many primes.

Proof:
Suppose that $p_1, p_2, ..., p_n$ were the only primes. Let M be the product of these primes. Since $M + 1 > 1$, then there exists a prime q , such that $q \mid M$. Since $(M, M + 1) = 1$, we know $q \nmid M$. Therefore $qneq = p_i$, for all i such that $1 \leq i \leq n$ contrary to hypothesis. So there exist infinitely many primes.

Chapter 2

CONTINUED FRACTIONS:-

Definition 2.1 *Let m be a non-zero real number. We define the continued fraction of m as follows*

$$m = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots\, a_{n-2} + \cfrac{b_{n-2}}{a_{n-1} + \cfrac{b_{n-1}}{a_n}}}}}}$$

where the a_i's and b_i's may be non-negative reals or complex, and a_0 may be zero.

- The numbers a_i are called partial quotients of the continued fraction.

- A simple continued fraction is one whereall the a_i's are positive integers with the exception of a_0 which may be negative or zero and all the b_i's equal to one.

that is $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \ddots + \ddots\, a_{n-2} + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}}$

If there exists such a_n at which the expansion terminates then it is called finite simple continued fractionion or terminating continued fraction. Otherwise it is said to be infinite. Denoted by $[a_0; a_1, ..., a_n]$

EXAMPLE:1

Take the rational number be $\frac{47}{17}$

$$47 = 2 \times 17 + 3$$

$17 = 1 \times 13 + 4$

$13 = 3 \times 4 + 1$

$4 = 4 \times 1 + 0$

$\frac{47}{17} = 2 + \frac{1}{1+\frac{1}{3+\frac{1}{4}}} = [2;1,3,4]$

$\frac{47}{17} = 2 + \frac{1}{1+\frac{1}{3+\frac{1}{3+\frac{1}{1}}}} = [2;1,3,3,1]$

So we noticed that the continued fraction expansion of a rational number is not unique.

EXAMPLE:2
Take the rational number$\frac{9}{7}$,

we can write it as a finite simple continued fraction as follows:-

$\frac{9}{7} = 1 + \frac{2}{7}$

$= 1 + \frac{1}{\frac{7}{2}}$

$= 1 + \frac{1}{3+\frac{1}{2}}$

$= [1;3,2]$

Definition 2.2 *For $n \leq m$, $c_n = [a_0; a_1, ..., a_n]$ is called the nth convergent of the continued fraction $[a_0; a_1, ..., a_m]$*

Theorem 2.1 *Let $a_0, a_1, ..., a_n$ be positive real numbers. Let*

$$p_0 = a_0, p_1 = a_1 a_0 + 1, p_{-1} = 1, p_n = a_n p_{n-1} + p_{n-2}$$

$$q_0 = 1, q_1 = a_1, q_{-1} = 0, q_n = a_n q_{n-1} + q_{n-2}$$

Then the n_{th} convergent is given by $c_n = \frac{p_n}{q_n}$

proof:

We proceed by induction. For n $= 1$,

$$[a_0; a_1] = a_0 + \frac{1}{a_1}$$

$$= \frac{a_1 a_0 + 1}{a_1}$$

$$= \frac{a_1 p_0 + p_{-1}}{a_1 q_0 + q_{-1}}$$

$$= \frac{p_1}{q_1}$$

as desired.

Now, suppose the theorem holds for n.

$$[a_0; ..., a_{n+1}] = [a_0; ..., a_{n-1}, a_n + \frac{1}{a_{n+1}}]$$

$$= \frac{(a_n + \frac{1}{a_{n+1}})p_{n-1} + p_{n-2}}{(a_n + \frac{1}{a_{n+1}})q_{n-1} + q_{n-2}}$$

$$= \frac{(a_{n+1} a_n + 1)p_{n-1} + a_{n+1} p_{n-2}}{(a_{n+1} a_n + 1)q_{n-1} + a_{n+1} q_{n-2}}$$

$$= \frac{a_{n+1} a_n p_{n-1} + a_{n+1} p_{n-2} + p_{n-1}}{a_{n+1} a_n q_{n-1} + a_{n+1} q_{n-2} + q_{n-1}}$$

9

$$= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}}$$

$$= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}}$$
$$= \frac{p_{n+1}}{q_{n+1}}$$

as desired.

EXAMPLE:

$\frac{19}{51} = [0; 2, 1, 2, 6]$

$$p_0 = 1, q_0 = 1$$

$$p_1 = 0 \times 2 + 1 = 1, q_1 = 2$$

$$p_2 = 1 \times 1 + 0 = 1, q_2 = 1 \times 2 + 1 = 3$$

$$p_3 = 2 \times 1 + 1 = 3, q_3 = 2 \times 3 + 2 = 8$$

$$p_4 = 6 \times 3 + 1 = 19, q_4 = 6 \times 8 + 3 = 51$$

$$\therefore c_0 = \frac{p_0}{q_0} = \frac{0}{2} = 0$$

$$c_1 = \frac{p_1}{q_1} = \frac{1}{2}$$

$$c_2 = \frac{p_2}{q_2} = \frac{1}{3}$$

$$c_3 = \frac{p_3}{q_3} = \frac{3}{8}$$

$$c_4 = \frac{p_4}{q_4} = \frac{19}{51}$$

Theorem 2.2 *For $1 \leq n \leq m$,*

If $c_n = \frac{p_n}{q_n}$ is the nth convergent of the finite simple continued fraction$[a_0; a_1, ..., a_n]$

then

(1)$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$; and

(2)$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$.

proof:-

(1) $p_n q_{n-1} - p_{n-1} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(a_n q_{n-1} + q_{n-2})$

$= p_{n-2} q_{n-1} + a_n p_{n-1} q_{n-1} - a_n p_{n-1} q_{n-1} - p_{n-1} q_{n-2}$

$=(-1)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$

$= ....$

$= (-1)^n (p_0 q_{-1} - p_{-1} q_0)$

$= (-1)^n (-1)$

$=(-1)^{n+1}$

$= (-1)^{n-1}$

as desired.

(2) $p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-2} - (a_n q_{n-1} + q_{n-2}) p_{n-2}$

$= a_n p_{n-1} q_{n-2} + p_{n-2} q_{n-2} - p_{n-2} q_{n-2} - a_n p_{n-2} q_{n-1}$

$=a_n(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$

$$= (-1)^{n-2} a_n$$

$$= (-1)^n a_n$$

as desired.

COROLLARY:-

For $1 \le k \le n$: p_k, q_n are relatively prime

    proof:-

$\because$ there is a linear combination of p_n and q_n that is equal to ± 1

    If $d = (p_k, q_k)$

then $d \mid (-1)^{k-1}$
    $\because d > 0 \Rightarrow d = 1$.

$\therefore p_k, q_n$ are relatively prime .

Now we will see how to solve the linear Diophantine equation $ax + by = c$ using continued fraction.

    Since, no solution of this equation exists if $d \nmid c$, where $d = (a, b)$, so there is no harm in assuming that $d \mid c$.

For;

if $(a, b) = d > 1$ then $\frac{a}{d} x + \frac{b}{d} y = \frac{c}{d}$.

Both equations have the same solution and in the latter case, we know that $\left(\frac{a}{d}, \frac{b}{d} \right) = 1$.
Notice that a solution of the equation $ax + by = c$ with $d = (a, b)$ may be obtained by solving first the Diophantine equation $ax + by = 1$ with $(a, b) = 1$

If integer x_0 and y_0 can be found for which $ax_0 + by_0 = 1$, then multiplying both sides by c we get,

$$a(cx_0) + b(cy_0) = c$$

Hence $cx_0 = x$ and $cy_0 = y$ is the solution of $ax + by = c$

To find a pair of integers x and y satisfying the equation $ax + by = 1$, we have to find the simple continued fraction expansion of the rational number $\frac{a}{b}$ where ,$\frac{a}{b} = [a_0; a_1, ..., a_n]$

The last two convergents of this continued fraction are

$$c_{n-1} = \frac{p_{n-1}}{q_{n-1}}$$

$$c_n = \frac{p_n}{q_n} = \frac{a}{b}$$

As $(p_n, q_n) = (a, b)$

we have $p_n = a$ and $q_n = b$

Then $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$

$$\implies aq_{n-1} - p_{n-1}b = (-1)^{n-1}$$

Thus with $x = q_{n-1}$ and $y = -p_{n-1}$

So $ax + by = (-1)^{n-1}$

If n is odd ,then $ax + by = 1$ has particular solution
$x_0 = q_{n-1}$ and $y_0 = -p_{n-1}$
If n is even ,then $ax + by = 1$ has particular solution
$x_0 = -q_{n-1}$ and $y_0 = p_{n-1}$

and the general solution is
$x = x_0 + bt$,$y = y_0 - at$;$t = \pm 1, \pm 2, ...$

EXAMPLE:-

solve the linear Diophantine equation

$$172x + 20y = 1000$$

by using simple continued fraction.

We have,$(172, 20) = 4$

$172x + 20y = 1000$

$\implies 43x + 5y = 250$

Firstly we have to find a particular solution to $43x + 5y = 250$

$\dfrac{43}{5} = [8; 1, 1, 2]$

The convergent are

$c_0 = \frac{8}{1}$, $c_1 = \frac{9}{1}$, $c_2 = \frac{17}{2}$, $c_3 = \frac{43}{5}$

$p_2 = 17, p_3 = 43$

$q_2 = 2, q_2 = 5$

Now $p_3 q_2 - q_3 p_2 = (-1)^{3-1}$
$\implies 43 \times 2 - 5 \times 17 = 1$

multiplying both sides by 250, we get
$43 \times 500 + 5(-4250) = 250$

So, the particular solution is given by $x_0 = 500, y_0 = (-4250)$
and the general solution is given by $x = 500 + 5t$, $y = -4250 - 43t$
$; t = 0, \pm 1, \pm 2 \pm, \ldots$

Periodic Continued Fraction:-

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}$$

$$= 1 + \frac{1}{2+\frac{1}{1+\sqrt{2}}}$$

$$= 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$$

$$\therefore \sqrt{2} = [1; 2, 2, 2, , \dots] = [1; \overline{2}]$$

$$\sqrt{3} = [1; \overline{1, 2}]$$

Here we see that partial quotients 2 and 1, 2 repeat indefinitely, such fractions are called periodic.

We write a periodic continued fraction as, $[a_0; a_1, \dots, a_m, b_1, b_2, \dots, b_n, \dots] = [a_0; a_1, \dots, a_m, \overline{b_1, b_2, \dots, b_n}]$

Here $b_1, b_2, \dots, b_n$, is the period of the expansion and the length of the period is n.

EXAMPLE:-

$\sqrt{14} = [3; \overline{1, 2, 1, 6}]$, is a continued fraction whose period 1,2,1,6 has length 4.

NOTE:-

If $[a_0; a_1, \dots]$ is an infinite continued fraction with positive terms, then its value is an irrational number.
Every irrational number x has a unique infinite continued fraction $[a_0; a_1, \dots]$ expansion whose terms are given recursively by

$$x_0 = x \text{ and } a_k = [x_k] , x_{k+1} = \frac{1}{x_k - a_k} , k \geq 0$$

EXAMPLE:-

$x = \sqrt{23} \approx 4.8$

$x_0 = 4 + (\sqrt{23} - 4)$; $a_0 = 4$

$x_1 = \frac{1}{x_0 - [x_0]} = 1 + \frac{\sqrt{23} - 3}{7}$; $a_1 = 1$

$x_2 = \frac{1}{x_1 - [x_1]} = 3 + \frac{\sqrt{23} - 4}{2}$; $a_2 = 3$

$x_3 = \frac{1}{x_2 - [x_2]} = 1 + \frac{\sqrt{23} - 3}{2}$; $a_3 = 1$

$x_4 = \frac{1}{x_3 - [x_3]} = 8 + (\sqrt{23} - 4)$; $a_4 = 8$

So $\sqrt{23} = [4; \overline{1, 3, 1, 8}]$

Chapter 3

PELL'S EQUATION:-

John Pell (1611-1685) was an English mathematician who taught mathematics in Holland, at the universities of Amsterdam and Breda in 1640s. Pell's equation has a long fascinating history. Its first recorded appearance is in the cattle problem of Archimedes. This problem involves eight different kinds of cattle and ask the reader to determine how many there are of each kind. After facing a lot of problems it finally reduced to solving the Pell's equation $X^2 - 4729494Y^2 = 1$. The first significant progress in solving the Pell's equation was made in India as early as A.D. 628, by Brahmagupta. Brahmagupta described how to use the known solution to a Pell's equation to create new solutions and Bhaskaracharya in 1150 A.D. gave a method of solving Pell's equation. The modern European history of Pell's equation begins in 1657 when Fermat challenged his fellow mathematician to solve the equation $X^2 - 61Y^2 = 1$ several of them found the smallest solution, which was $(X, Y) = (1766319049, 226153980)$ and William Brouncker in described a general method for solving Pell's equation. Brouncker found the solution $(32188120829134849, 181938015856410)$.

J. Wallis described Brounckers method in a book on algebra and number theory and Wallis and Fermat both asserted that the Pell's equation always has a solution. Euler mistakenly thought that the method in Wallis book was due to John Pell, and so Euler assigned the equation the name Pell's equation. But John Pell has nothing to do with the so called Pell's equation .

Definition 3.1 *The quadratic Diophantine equation of the form $x^2 - dy^2 = \pm 1$ where d is a positive square free integer is called a Pell's equation.*

Problems leading to Pell's equation:

One of the main reasons for the popularity of Pell's equation is the fact that many natural questions that one might ask about integer's leads to a quadratic equation in two variables, which can be casted as a Pell's equation .

Square Triangular Numbers:

The numbers which can be arranged in a shape of triangle is called triangular numbers, whereas the numbers which can be arranged in a shape of square are called square numbers.

1+2=3

1+2+3=6
(Triangular Numbers)

1+2+3+4=10

2^2=4

3^2=9
(Square Numbers)

4^2=16
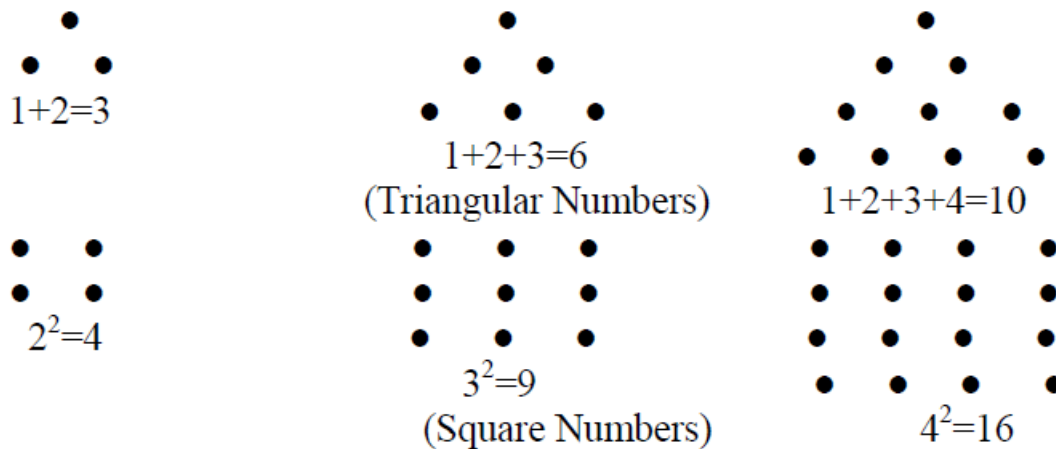
The mth triangular number is $= \frac{m(m+1)}{2}$ and the nth square number $= n^2$

We observe that the sum of two adjacent triangular numbers is square.

But what about the case when an individual triangular number is square?

So, the square triangular numbers are solution to the equation

$n^2 = \frac{m(m+1)}{2}, m, n \in Z^+$

Multiplying both sides by 8 we get

$8n^2 = 4m^2 + 4m$

$= (2m + 1)^2 - 1$

Leting $x = 2m + 1$, $y = 2n$, we have

$2y^2 = x^2 - 1$

$\implies x^2 - 2y^2 = 1$

Solution to $x^2 - 2y^2 = 1$ gives square-triangular numbers with $m = \frac{x-1}{2}$ and $n = \frac{y}{2}$

BRAHMAGUPTA'S METHOD:

$nx^2 + 1 = y^2$ (Pell's equations) leading to $y^2 - nx^2 = 1$

Identify:- $(b^2 - na^2)(d^2 - nc^2) = (bd \pm nac)^2 - n(bc \pm ad)^2$

From this we observe that if $(b^2 - na^2)$ and $(d^2 - nc^2)$ are both 1, then $(bd \pm nac)^2 - n(bc \pm ad)^2 = 1$, i.e. if (a, b) and (c, d) are solution to pell equation then $(bc \pm ad, bd \pm nac)$ is also a solution.

Brahmagupta's Lemma:-

If(a, b) and (c, d) are integer solution of pell type equation of the form $na^2 + k = b^2$ and $nc^2 + k' = d^2$, then $(bc \pm ad, bd \pm nac)$ are both integer solution of Pell type equation $nx^2 + kk' = y^2$, using the method of composition, if (a, b) satisfies pell's equation, then so does $(2ab, b^2 + na^2)$ which is obtained by composing (a, b) with itself. Another solution can be obtained by composing (a, b) with $(2ab, b^2 + na^2)$.

SOLVING PELL'S EQUATION USING BRAHMAGUPHA'S METHOD:-

By Brahmagupta's lemma if (a, b) is solution of $nx^2 + k = y^2$,

then composing (a, b) with itself gives us $(2ab, b^2 + na^2)$ as a solution of $nx^2 + k = y^2$ and dividing by k ,we get

$$x = \frac{2ab}{k} \text{ and } y = \frac{b^2 + na^2}{k}$$

Which is a solution to the Pell's equation $nx^2 + 1 = y^2$.

For most values of k this idea is not helpful because x, y are not integers, but when k is $\pm 1, \pm 2, or \pm 4$ this idea helps a lot.

When $k = 2$, as (a, b) is solution of $nx^2 + k = y^2$ then, $na^2 = b^2 - 2$, so,$x = \frac{2ab}{2} = ab$, $y = \frac{b^2 + na^2}{2} = \frac{b^2 + b^2 - 2}{2} = \frac{2b^2 - 2}{2} = b^2 - 1$.

For $k = -2$,$na^2 = b^2 + 2$ and $x = \frac{2ab}{-2}$,$y = \frac{b^2 + b^2 + 2}{2} = -b^2 - 1$.

For $k = 4, -4$,we can get solutions to the Pell's equation by method of composing, but it's too much complicated. So Brahmagupta was able to show that if we can find (a, b) which nearly satisfies Pell's equation in the sense $na^2 + k = b^2$ where $k = \pm 1, \pm 2, \pm 4$ then we can find many integer

20

solutions to Pell's equation.

EXAMPLE:-
Brahmagupta's solution of the Pell's equation

$$83x^2 + 1 = y^2$$

Here $a = 1$, $b = 9$ satisfy the equation $83 \times 1^2 - 2 = 9^2$.

So applying the above method, we find that,

$$x = \frac{2ab}{k} \text{ and } y = \frac{b^2 + na^2}{k}$$

, is a solution to $83x^2 + 1 = y^2$

i.e; $x = \frac{2 \times 9}{2} = 9$, $y = \frac{81 + 83 \times 1}{2} = 82$

i.e; $(9, 82)$ is a solution.

Applying method of composition to $(9, 82)$ and $(9, 82)$
we get $(2 \times 9 \times 82, 82 \times 82 + 83 \times 81) = (1476, 13447)$

Again applying method of composition to $(9, 82)$ and $(1476, 13447)$. we have,
$x = 9 \times 13447 + 82 \times 1476 = 242055$
$y = 82 \times 13447 + 83 \times 9 \times 1476 = 2205226$.

By applying again and again the method of composition, we can generate further solutions (x, y).

Results on Pell's Equation:-

The Pell equation is a Diophantine equation of the form $x^2 - dy^2 = 1$. Given d, we want to find all integer pairs (x, y) that satisfy the equation. Since any solution (x, y) yields multiple solution $(\pm x, \pm y)$, we restrict ourselves to those solutions where x and y are positive integers.

We generally take d to be a positive non-square integer; otherwise there are only uninteresting solutions. If $d < 0$, then $(x, y) = (\pm 1, 0)$, in the case $d < -1$ and $(x, y) = (0, \pm 1)$ or $(\pm 1, 0)$ in the case $d = -1$, if $d = 0$, then $x = \pm 1$ (y arbitrary) and if d is non zero square ,then dy^2 and x^2 are consecutive squares, implying that $(x, y) = (\pm 1, 0)$.

Notice that the Pell equation always has the trivial solution $(x, y) = (1, 0)$

The following result is well known:
If $\frac{p_n}{q_n}$ as the nth converget to the irrational number x ,
then $\mid x - \frac{p_n}{q_n} \mid < \frac{1}{q_{n+1}q_n} < \frac{1}{q_n{}^2}$

Theorem 3.1 *If $\frac{p}{q}$ is a convergent of the continued fraction expansion of $\sqrt{d}$ then,$x = p, y = q$ is a solution of the equation $x^2 - dy^2 = k$ where $\mid k \mid < 1 + 2\sqrt{d}$*

PROOF:- If $\frac{p}{q}$ is a convergent of $\sqrt{d}$,then
$\mid \sqrt{d} - \frac{p}{q} \mid < \frac{1}{q^2}$

$\therefore \mid p - q\sqrt{d} \mid < \dfrac{1}{q}$

Now $\mid p + q\sqrt{d} \mid$
$= \mid (p - q\sqrt{d}) + 2q\sqrt{d} \mid \leq \mid p - q\sqrt{d} \mid + \mid 2q\sqrt{d} \mid < \dfrac{1}{q} + 2q\sqrt{d} < (1 + 2\sqrt{d})q$

These two inequalities combine to yield
$\mid p^2 - dq^2 \mid = \mid p - q\sqrt{d} \mid \mid p + q\sqrt{d} \mid < \frac{1}{q}(1 + 2\sqrt{d})q = 1 + 2\sqrt{d}$

EXAMPLE:-
Let $d = 7$
$\sqrt{7} = [2; \overline{1, 1, 1, 4}]$

The first two convergent of $\sqrt{7}$ are $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, ...$

Now calculating $p_n{}^2 - 7q_n{}^2$, we find that
$2^2 7 \times 1^2 = -3$
$3^2 7 \times 1^2 = 2$
$5^2 7 \times 2^2 = -3$
$8^2 7 \times 3^2 = 1$
Hence $x = 8, y = 3$, provides a positive solution of $x^2 - 7y^2 = 1$

⋆ If d is a positive integer that is not a perfect square, then the continued fraction expansion of $\sqrt{d}$ necessarily has the form $\sqrt{d} = [a_0; \overline{a_1, a_2, a_3, ..., a_3, a_2, a_1, 2a_0}]$
.

EXAMPLE:-
$\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$

$\sqrt{73} = [8; \overline{1, 1, 5, 5, 1, 1, 16}]$

⋆
If the length of the period of the continued fraction expansion of $\sqrt{d}$ is n then the fundamental solution of the equation $x^2 - dy^2 = 1$ is given by $x = p_{n-1}, y = q_{n-1}$ when n is even and by $x = p_{2n-1}, y = q_{2n-1}$ when n is odd.

Finding the fundamental solution may become a difficult task, as the value of the fundamental solution may be very large, even for comparatively small values of d .

For example the equation $x^2 - 991y^2 = 1$ has the fundamental solution $x = 379516400906811930638014896080$, and $y = 12055735790331359447442538767$.

The solution is even worse with $x^2 - 1000099y^2 = 1$, where the smallest positive integer has 1118 digits.

So, everything depends upon the continued fraction expansion of $\sqrt{d}$ and in case of $\sqrt{1000099}$, the period has 2174terms.

There might be possibilities that the solution of $x^2 - dy^2 = 1$ may be small for a given value of d and very large for succeeding values of d.

For example the equation $x^2 - 67y^2 = 1$, whose fundamental solution is given by , $x = 48842$,5967

But with the case $d = 66$ where the solution is $x == 65$ and $y = 8$

And with $d = 68$,where the solution is $x = 33$,$y = 4$

$\star$ From any solution of $x^2 - dy^2 = 1$, we can obtain infinitely many solutions.

Let (x_1, y_1) be a solution of $x^2 - dy^2 = 1$. Then we can generate another solution by the following process:

$$x_1^2 - dy_1^2 = 1$$

$$(x_1 + y_1\sqrt{d})(x_1 - y_1\sqrt{d}) = 1$$

$$(x_1 + y_1\sqrt{d})^2(x_1 - y_1\sqrt{d})^2 = 1$$

$$(x_1^2 + 2x_1y_1\sqrt{d} + dy_1^2)(x_1^2 - 2x_1y_1\sqrt{d} + dy_1^2) = 1$$

$$(x_1^2 + dy_1^2 + 2x_1y_1\sqrt{d})(x_1^2 + dy_1^2 - 2x_1y_1\sqrt{d}) = 1$$

$$(x_1^2 + dy_1^2)^2 - d(2x_1y_1)^2 = 1$$

We see that we have got a equation which is nothing but a Pells Equation and $(x_1^2 + dy_1^2, 2x_1y_1)$ is a solution. Applying the process repeatedly we can

24

get as many solution as we desire.

Alternatively, suppose we consider integer powers of $(x_1 + y_1\sqrt{d})$

$$(x_1 + y_1\sqrt{d})^n = x_1^n + \binom{n}{1}\sqrt{d}x_1^{n-1} + \binom{n}{2}dx_1^{n-2}y_1^2 + \binom{n}{3}d^{\frac{3}{2}}x_1^{n-3}y_1^3 + ... + d^{\frac{n}{2}}y_1^n$$
$$= (x_1^n + \binom{n}{2}dx_1^{n-2}y_1^2 + ...) + \sqrt{d}(\binom{n}{1}x_1^{n-1} + \binom{n}{3}x_1^{n-3}y_1^3 + ...)$$
$$= x_n + y_n d$$

Theorem 3.2 *Let (x_1, y_1) be the fundamental solution of $x^2 - dy^2 = 1$ .Then every pair of integers (x_n, y_n) defined by the condition*
$x_n + y_n\sqrt{d} = (x_1 - y_1\sqrt{d})^n$ *, is also a positive solution, $n = 1, 2, 3, ...$*

PROOF:-

$$x_n + y_n\sqrt{d} = (x_1 - y_1\sqrt{d})^n$$

Further, because x_1 and y_1 are positive x_n and y_n are both positive integers. Since is a solution of $x^2 - dy^2 = 1$, we have
$x_n^2 - dy_n^2 = (x_n + y_n\sqrt{d})(x_n - y_n\sqrt{d}) = (x_1 + y_1\sqrt{d})^n(x_1 - y_n\sqrt{d})^n$
$= (x_1^2 - dy_1^2)^n = 1$

Hence (x_n, y_n) is a solution.

EXAMPLE:-
$x^2 - 35y^2 = 1$

$x_1 = 6, y_1 = 1$ forms the fundamental solution.
Now
$$x_2 + y_2\sqrt{35} = (6 + \sqrt{35})^2 = 71 + 12\sqrt{35}$$

So, $x_2 = 71, y_2 = 12$

$\implies (x_2, y_2) is a solution$

$$\because 71^2 - 35 \times 12^2 = 5041 - 5040 = 1$$

$$x_3 + y_3\sqrt{35} = (6 + \sqrt{35})^3 = 846 + 143\sqrt{35}$$

So, $x_3 = 846, y_3 = 143$

$\implies (x_3, y_3) is a solution$
$\because 846^2 - 35 \times 143^2 = 715716 - 715715 = 1$

In this way we can generate infinitely many solutions.

$\star$ If (x_1, y_1) is the fundamental solution of $x^2 - dy^2 = 1$, then the solutions (x_n, y_n) is also given by

$$x_n = \tfrac{1}{2}(x_1 + y_1\sqrt{d})^n + \tfrac{1}{2}(x_1 - y_1\sqrt{d})^n$$

$$y_n = \tfrac{1}{2\sqrt{d}}(x_1 + y_1\sqrt{d})^n - \tfrac{1}{2\sqrt{d}}(x_1 - y_1\sqrt{d})^n$$

Theorem 3.3 *If $x_1 > 1$, $y_1 \geq 1$ and $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$, then $x_{n+1} > x_n$ and $y_{n+1} > y_n$ for positive n*

PROOF:-
We will prove this by induction, observe that
$x_2 = x_1^2 + dy_1^2$ and $y_2 = 2x_1 y_1$

Since $x_1 > 1$, $y_1 \geq 1$ and d is a positive integer it is clear that $x_2 > x_1$, $y_2 > y_1$.
So the result is true for n=1

Now assume the solution (x_n, y_n) with x_n and y_n positive integers greater than 1. We have,

$$(x_{n+1} + y_{n+1}\sqrt{d}) = (x_1 + y_1\sqrt{d})^{n+1}$$
$$= (x_1 + y_1\sqrt{d})(x_1 + y_1\sqrt{d})^n$$
$$= (x_1 + y_1\sqrt{d})(x_n + y_n\sqrt{d})$$
$$= (x_1 x_n + dy_1 y_n) + (x_1 y_n + x_n y_1)\sqrt{d}$$

$$\therefore x_{n+1} = (x_1 x_n + dy_1 y_n)$$
and $y_{n+1} = (x_1 y_n + x_n y_1)$

We know $x_1 x_n > x_n$ and $dy_1 y_n > 0$

$$\implies x_{n+1} = x_1 x_n + dy_1 y_n > x_n$$

$x_1 y_n > y_1$ and $x_n y_1 > o$

$$\implies y_{n+1} = x_1 y_n + x_n y_1 > y_n$$

$\therefore$ We have $x_{n+1} > x_n$ and $y_{n+1} > y_n$

Definition 3.2 *Associated Pell's Equation:*

The equation $x^2 - dy^2 = -1$ is called associated Pells equation or, negative Pells equation.

Let t be the length of the period the continued fraction expansion of $\sqrt{d}$. If t is even, then $x^2 - dy^2 = -1$ has no solution. If t is odd, then $x^2 - dy^2 = -1$ has infinitely many solutions and all given by
$$(x_n, y_n) = (p_{(2n-1)(t-1)}, q_{(2n-1)(t-1)}).$$

Example:
Consider the equation $x^2 - 7y^2 = -1$
$\because \sqrt{7} = [2; \overline{1, 1, 1, 4}]$ so $t = 4$
$\therefore$ It has no solution.

Consider the equation $x^2 - 41y^2 = -1$

$\because \sqrt{41} = [6; \overline{2, 2, 12}]$ so $t = 3$

$\therefore$ it has solution . The first two such solutions are
$(x_1, y_1) = (p_2, q_2) = (32, 5)$
$(x_2, y_2) = (p_8, q_8) = (131168, 20485)$

Definition 3.3 *Pell numbers:*

The Pell number is defined by the recurrence relation

$$P_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ 2P_{n-1} + P_{n-2} & \text{otherwise.} \end{cases}$$

That is, the Pell numbers sequence starts with 0 and 1, and then each Pell number is the sum of twice the previous Pell number and the Pell number before that.

The first few terms of the sequence are 0, 1, 2, 5, 12 , 29 ,70, 169 , 408 ,985, 2378, ...

The Pell number can also be expressed by the closed form

$$p_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

Definition 3.4 *Pell prime:-*
A Pell prime is a Pell number that is prime. The first few Pell primes are 2,5,29 . A Pell number can only be prime if itself is prime

28

Definition 3.5 *Pell-Lucas numbers:-*

The Pell-Lucas numbers are defined by the recurrence relation

$$
Q_n = \begin{cases} 2 & \text{if } n = 0; \\ 2 & \text{if } n = 1; \\ 2Q_{n-1} + Q_{n-2} & \text{otherwise.} \end{cases}
$$

That is, the first two numbers in the sequence are both 2, and each successive number in formed by adding twice the previous Pell-Lucas number to the Pell-Lucas number before that, or, by adding the next Pell number to the previous Pell number.

The first few terms of the sequence are 2,2,6,14,34,82,198,478.

The Pell-Lucas numbers can be expressed by the closed form
$Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n$

The Pell-Lucas numbers are all even

Definition 3.6 *Half companion Pell numbers:-(H_n)*

$$
H_n = \begin{cases} 1 & \text{if } n = 0; \\ 2P_{n-1} + H_{n-2} & \text{otherwise.} \end{cases}
$$

$$
P_n = \begin{cases} 0 & \text{if } n = 0; \\ P_{n-1} + H_{n-1} & \text{otherwise.} \end{cases}
$$

Chapter 4

APPLICATIONS OF PELL'S EQUATION:-

1.Double equations:

Find $t \in Z$ such that

$10t + 9 = x^2$ and
$5t + 4 = y^2$

Ans: We have $t = \frac{x^2 - 9}{10} = \frac{y^2 - 4}{5}$

$\implies 5(\frac{x^2 - 9}{10}) + 4 = y^2$

$\implies x^2 - 2y^2 = 1$ which is a Pell's equation with $d = 2$

Solutions are listed in the following table:-

x	3	17	99	577	3363	19601
y	2	12	70	408	2378	13860
t	0	28		33292	1130976	

2. Rational approximation of square roots:-

We cannot write $\sqrt{d} = \frac{x}{y}$, $x, y \in Z$, $\sqrt{d}$ is irrational

But $x^2 - dy^2 = 1$

$\implies \frac{x^2}{y} = d + \frac{1}{y^2} \approx d$

so, Pell solutions lead to good rational approximation to $\sqrt{d}$.

EXAMPLE:-
fourth solution to $x^2 - 2y^2 = 1$ is $(x, y) = (577, 408)$ and $\frac{577}{408} = 1 \times 4142156$
while $\sqrt{2} = 1.4142135$

3. Sum of consecutive integers:-

EXAMPLE:-
1+2=3
1+2+3+...+14=15+16+...+20 In general,if we have
$1 + 2 + ... + k = (k + 1) + ... + l,$
then $\frac{(1+k)k}{2} = \frac{(k+1+l)(l-k)}{2}$
$2k^2 + 2k = l^2 + l$
and $2((k + \frac{1}{2})^2 - \frac{1}{4}) = (l + \frac{1}{2})^2 - \frac{1}{4}$ which finally simplifies to
$(2l + 1)^2 - 2(2K + 1)^2 = -1$ which is a associated Pells equation and can be
written as $x^2 - 2y^2 = -1, x, y > 0 : x = 2l + 1, y = 2k + 1$
$x^2 - 2y^2 = -1$,both x and y odd

x	1	7	41	239	1393	8119
y	1	5	29	169	985	5741
K=(y-1)/2	0	2	14	84	492	2870
l=(x-1)/2	0	3	20	119	696	4059

so $1 + 2 + ... + 84 = 85 + ... + 119$

Pythagorean triangle with consecutive legs like (3,4,5) ,(20,21,29)...:-

In general we are interested in solving $m^2 + (m + 1)^2 = n^2$

31

Clearly n^2 is odd, so n is odd. $2m^2 + 2m + 1 = n^2$

$\implies 2(m^2 + m) + 1 = n^2$

$\implies 2((m + \frac{1}{2})^2 - \frac{1}{4}) + 1 = n^2$

which gives

$(2m + 1)^2 + 1 = 2n^2$

and finally $(2m + 1)^2 - 2n^2 = -1$

which is an associated Pells equation

So $m^2 + (m + 1)^2 = n^2 \iff (2m + 1)^2 - 2n^2 = -1$

The solutions are listed in the following table.

x	1	7	41	239	1393	8119
y	1	5	29	169	985	5741
m=(x-1)/2	0	3	20	119	696	4059
n=y	1	5	29	169	985	5741

$$3^2 + 4^2 = 5^2$$
$$20^2 + 21^2 = 29^2$$
$$119^2 + 120^2 = 169^2$$

Consecutive Heronian triangles:-

EXAMPLE:
The 3,4,5 right triangle has area 6

$A = \sqrt{s(s - a)(s - b)(s - c)}; s = \frac{a+b+c}{2}$

Find Heronian triangle with consecutive sides $a - 1, a, a + 1$ and thus $s = \frac{3a}{2}$

$A^2 = \frac{3a}{2} \cdot \frac{a}{2} \cdot \frac{a+2}{2} \cdot \frac{a-2}{2}$

$\implies 4A^2 = 3a^2(a^2 - 4)$

So $A^2 = 3x^2(x^2 - 1)$

unique factorization $\implies x^2 - 1 = 3y^2$
$\implies x^2 - 3y^2 = 2$;a=2x and A=3xy

x	2	7	26	97	362	1351
y	1	4	15	56	209	780
a	4	14	52	194	724	2702
A	6	84	1170	16926	226974	3161340

So, we have triangles with sides 3,4,5(area=6);13,14,15 (area==84); 51,52,53(area=1170) and so on.

CONCLUSION:

Over the course of this project , we have investigated simple continued fractions and explored an application of solving Pell's equation.If $\frac{p}{q}$ is a convergent of the continued fraction expansion of $\sqrt{d}$ then,$x = p, y = q$ is a solution of the equation $x^2 - dy^2 = k$ where $\mid k \mid < 1 + 2\sqrt{d}$

REFERENCES:

[1] David M. Burton, Elementary Number Theory, Tata McGraw-Hill, Seventh Edition (2016).

[2]Jorn Steuding, Diophantine Analysis, Chapman and Hall/CRC, (2005).

[3]Titu Andreescu and Dorin Andrica,An Introduction to Diophantine Equations. Gil Publishing House, (2002).

[4] Jesse Unger ,Solving Pelt's Equation with Continued Fractions