INTRODUCTION TO L(2,1) LABELLING OF GRAPHS

A Dissertation submitted in partial fulfillment of the

requirement for the award of

DEGREE OF MASTER OF SCIENCE

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By

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CERTIFICATE

This is to certify that the dissertation entitled "INTRODUCTION TO L(2,1) LABELLING OF GRAPHS" is a bonafide record of the work done by AISWARYA P PREMGI under my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Ms Anna Treesa Raj , Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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I making this project not only for marks but to also increase my knowledge .

Thanks again to all who help.

ERNAKULAM — AISWARYA P PREMGI

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CONTENTS

INTRODUCTION

The study of the distance two labelling of graphs is motivated from the channel assignment problem introduced by Hale.
The channel assignment problem is the assignment of frequencies to television and radio transmitters subject to restrictions imposed by the distance between transmitters. This problem was first formulated as a graph coloring problem by Hale.
In 1988,Roberts (in a private communication with Griggs) proposed a variation of the channel assignment in which close transmitters must receive different channels and very close transmitters must receive channels atleast two apart.
Motivated by this variation, Griggs and Yeh first proposed and studied the L(2,1) labelling of a simple graph with a condition at distance two.

The mathematical modelling of channel assignment problem is as follows.

To convert the channel assignment problem into graph theory, the transmitters are represented by the vertices of a graph. Two vertices x and y are very close if the distance between them is one and close if the distance between x and y is two.
we denote d(x,y) to represent the shortest distance between the vertices x and y.
L(2,1) labelling was subsequently generalised to L(p,q) labelling problems of graph where p and q are non-negative intergers.
L(p,q) labelling is also known as L(p,q) coloring is infact a proper coloring of graph.

PRELIMINARIES

GRAPH: A graph is a pair of sets (V,E), where V is the set of vertices and E is the set of edges, formed by pair of vertices.

NEIGHBOURHOOD: A vertex v is a neighbhourhood of u in G if uv is an edge of G and u $\neq$ v

ADJACENT VERTICES: Two vertices u and v are adjacent if and only if there is an edge of G with u and v as its ends.

SIMPLE GRAPH: A graph with no loops and parallel edges are called simple graph.

COMPLETE GRAPH: A simple graph G is said to complete if its each pair of distinct vertices is joined by an edges. A complete graph with n vertices is denoted by K_n

DEGREE: The degree of a vertex v, denoted $\Delta(v)$ is the number of edges incident to it.Maximum degree of a graph G, denoted by $\Delta(G)$,is the greatest $\Delta(v)$ over all v$\in$ V(G)

PATH: A walk is called a path if all vertices are distinct. Where a walk is denoted by W = $v_0e_1v_1e_2v_2...e_kv_k$

BI-PARTITE GRAPH: A graph is Bi-partite if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge has one end in X and other end in Y.

K-PARTITE GRAPH: A K-partite graph is a graph whose graph vertices can be partitioned into K disjoined sets so that no two vertices within the same set are adjacent.

COMPLEMENT: Complement of a simple graph G = (V,E) is the simple graph $\overline{G}$=(V,$\overline{E}$) where the edges in $\overline{E}$ are exactly the edges not in G.

SUBGRAPH: A graph H is a subgraph of G (H$\subseteq$ G) if V(H)$\subseteq$V(G), E(H) $\subseteq$ V(G).

CYCLE: A cycle is a closed trail in which all the vertices are distinct

CONNECTED GRAPH: A graph is connected if it contains a u,v- path whenever u,v$\in$ V(G).

TREE: A tree is a connected graph containing no cycles.

HAMILTONIAN PATH:Hamiltonian path is a path in an undirected or directed graph that visits each vertex exactly once.

STABLE SET:An independent set or stable set is a set of vertices in a graph,no two of which are adjacent.

k-VERTEX COLOURING: k-vertex colouring of G is an assignment of k colours 1,2,3...k to the vertices of G.

PROPER k-VERTEX COLOURING:The colouring is proper if no 2 distinct adjacent vertices have the same colour.

k-VERTEX COLOURABLE GRAPH: A graph G is k-vertex colourable if G has a proper k-vertex colouring.

CHROMATIC NUMBER: The chromatic number χ of G that is χ(G) is the minimum k for which G is k-colourable.

BROOK'S THEOREM: If G is a connected graph other than a complete graph or an odd cycle,then χ(G) $\leqslant \Delta$ (G)

PIGEONHOLE PRINCIPLE: If we put more than n objects into n boxes then there is a box containing atleast $[\frac{n}{k}]$ objects.

DIAMETER OF A GRAPH: It is the maximum eccentricity of any vertex in the graph.That is, it is the greatest diastance between any pair of vertices.

CHAPTER-1

LABELLING NUMBERS OF SPECIAL CLASSES OF GRAPHS

1.1 L(p,q) LABELLING

L(p,q) labelling of a graph G=(V,E) is a function f from the vertex set to the positive integers suh that $|f(x) - f(y)| \geqslant$ p if d(x,y) = 1 and $|f(x) - f(y)| \geqslant$ q if d(x,y) = 2 ,where d(x,y) is the distance between the two vertices x and y in the graph G.

The major problem of interest with L(p,q) labelling concerns spans.

DEFINITION

SPAN

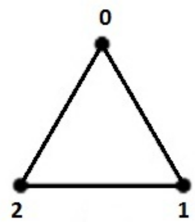
The span of an L(p,q) labelling f which is the difference between the largest and the smallest labels of f plus 1.
The $\lambda_{p,q}$ number of G is $\lambda_{p,q}$(G), the minimum span over all L(p,q) labelling of G.

EXAMPLES

1. Consider p=q=1 we get L(1,1) labelling.

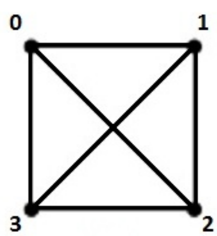
(a) COMPLETE GRAPH

(i)Consider K_3



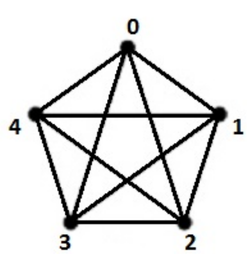
$\lambda(K_3) = 2$ = degree of K_3 = 3-1

(ii)Consider K_4



$\lambda(K_4) = 3$ = degree of K_4 = 4-1

(iii)Consider K_5



$\lambda(K_5) = 4$ = degree of $\lambda(K_5)$ = 5-1

In general we can say that for a complete graph K_n , minimum span $\lambda_{1,1}$ is the degree of the graph K_n = n-1.

(b) PATH

(i)Consider P_2

$\lambda(P_2) = 1$

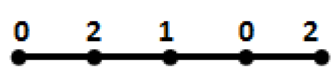
(ii) Consider P_3

$\lambda(P_3) = 2$

(iii)Consider P_4

$\lambda(P_4) = 2$

(iv) Consider P_5



$\lambda(P_5) = 2$

In general we can say that $\lambda(P_n)$ follows a sequence (0,2,1,0,2,1...)
Therfore $\lambda(P_n) = 2$

2. When p=1 and q=0 we get L(1,0) labelling which is a the usual proper coloring. That is, in L(1,0) labelling no 2 adjacent vertices have same label.

Most of the interest in L(p,q) labelling has been in the case where p=2 and q=1

1.2 L(2,1) LABELLING

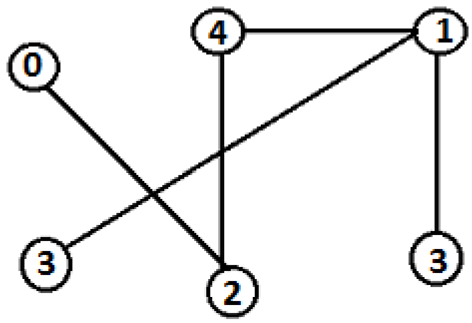
DEFINITION

An L(2,1) labelling of a graph G is a function f from the vertex set V(G) to the set of all non negative integers such that $|f(x) - f(y)| \geqslant 2$ if d(x,y)=1 and $|f(x) - f(y)| \geqslant 1$ if d(x,y) =2 where d(x,y) denotes the distance between x and y in G. A k-L(2,1) labelling is an L(2,1) labelling such that no label is greater than k.

DEFINITION

L(2,1) labelling number
The L(2,1) labelling number of G denoted by λ (G) , is the smallest number k such that G has a k-L(2,1) labelling.
In other words , λ (G) of G is the smallest number k such tat G has an L(2,1) labelling with max $\{f(v):v\epsilon V(G)\} = k$.

EXAMPLE



$\lambda = 4$

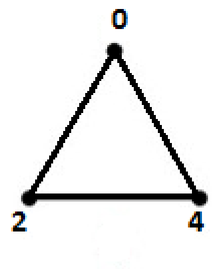
1.3 LABELLING NUMBERS OF SPECIAL CLASSES OF GRAPHS

1. $\lambda(K_n)$ = 2n-2, where K_n is a complete graph.

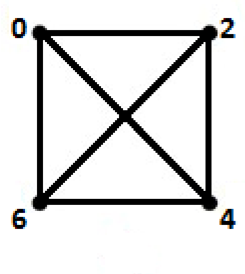
Proof:

Given K $_n$ with vertices v_1,v_2,v_3,...v_n.
f:V(G)$\longrightarrow \{0, 1, 2, ..., 2n - 2\}$ defined by f(v_i) = 2i-2 is a labelling of K_n.
So λ (K_n) $\leqslant$ 2n-2.
Claim: We can't label K_n with just the numbers 0,1,2...2n-3.
Note that we have the 2n-2 labels that need to be assigned to n vertices.
We can think this as n-1 disjoint pairs of consecutive labels in which n vertices must be placed.
By pigeonhole principle, one of these pairs of consecutive labels must contains 2 vertices.However, since these 2 vertices are adjacent in K_n , this violates our labelling condition.
Thus $\lambda(K_n)$ = 2n-2

Example



$\lambda(K_3)$=2×3-2=4



$\lambda(K_4)$ = 2×4-2=6

2. PATH

(i) $\lambda(P_2) = 2$

Proof:

First consider P_2.
We start by labelling one vertex 0.This forces the other vertex to be atleast 2.
So $\lambda(P_2) = 2$

(ii) $\lambda(P_3) = 3$

Proof:

For P_3, we can label the leftmost vertex 0, the middle vertex 3 and the right vertex 1
So $\lambda(P_3) \leqslant 3$

Claim: We can't label P_3 with just the number 0,1,2.

The label 1 would not be used anywhere or else it would have adjacent to 0,1,2 all of which violates adjacency rule. This leaves us with two labels (0 and 2) that must be assigned to three vertices.By the pigeonhole principle two of these vertices must receive the same label which necessarily violates the condition.

Before consider P_4, we need the following lemma.

LEMMA If H is a subgroup of G, then $\lambda(H) \leqslant \lambda(G)$.

Proof:

Let $\lambda(G) = m$ with corresponding labelling f: $V(G) \longrightarrow \{0, 1, 2...m\}$
Then g: $V(H) \longrightarrow \{0, 1, 2...m\}$ defined by $g(v) = f(v)$ $\forall$ v$\in$ V(H), is a labelling of H that uses no label greator than m.
Thus $\lambda(H) \leqslant \lambda(G)$.
The idea is,we can use the same labels we use on G to label the corresponding vertices of H.

(iii) $\lambda(P_4) = 3$

Proof:

Since P_3 is a subgraph of P_4, from our previous lemma,$\lambda(P_4) \geqslant \lambda(P_3) = 3$
The following figure shows we can label P_4 with no label greater than 3.

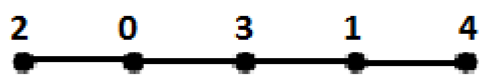
Thus $\lambda(P_4) \leqslant 3$ and the result follows.

(iv) $\lambda(P_n)$=4 for n$\geqslant$ 5

Proof:

First we show $\lambda(P_5) = 4$

The following figure shows we can label P_5 with no label greater than 4. So $\lambda(P_5) \leqslant 4$.



Claim: We can't label P_5 with just the numbers 0,1,2 and 3.

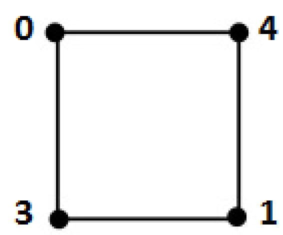
The labels 1 and 2 cannot be assigned to any independent vertex without violating either the adjacency rule or the distance 2 rule.
To see this, suppose one of the non-independent vertices of P_5 were labelled 1.Then only the label 3 can be assigned to its neighbour without violating the adjacency rule.
However, if both its neighbours receive the label 3, the distance two rule is violated.
So this, leaves us with 2 labels (0,3) that must be assigned to the three non-independent vertices..Again by the pigeonhole principle, two of these vertices must receive the same label, which necessarily violates the condition,So $\lambda(P_5)$ = 4.
Next we show $\lambda(P_5) = 4$ for n$>$ 5
Let P_n be a path with more than 5 vertices.
Since P_5 is a subgraph of P_n , $\lambda(P_n) \geqslant \lambda(P_5) = 4$.
Notice that we can cyclically repeat the label in P_5 (2,0,3,1,4,2,0,3...) and still get a proper labelling for any P_n.Thus $\lambda(P_n) \leqslant 4$ and the result follows.

3. CYCLES

$\lambda(C_n) = 4$ for n $\geqslant$ 3

Proof:

Since $C_3 = K_3$,we have $\lambda(C_3) = \lambda(K_3) = 4$.
Now consider C_4.
The following figure shows we can label C_4 with no label greater than 4.



so $\lambda(C_4) \leqslant 4$

Claim: We can't label C_4 with just the numbers 0,1,2 and 3.

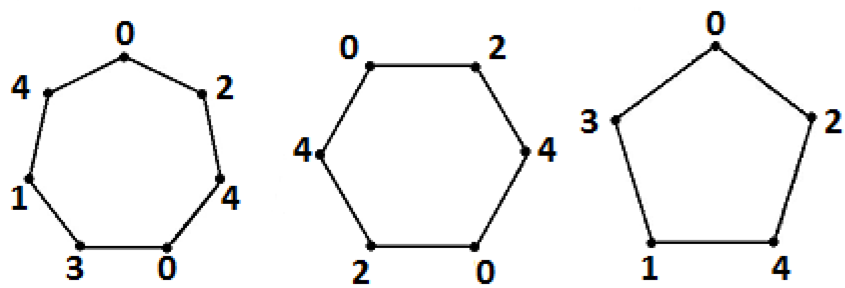
Since every vertex in C_4 is adjacent to two other vertices,we cannot use label 1 and 2 without violating the rules.

This leaves us with two labels (0 nd 3) that must be assigned to the four vertices.

Again by pigeonhole principle,two of these vertices must receive the same label, which necessarily violates the condition since any apart.
So $\lambda(C_4) = 4$

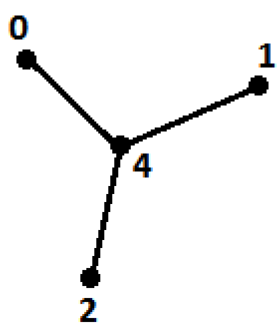
Now consider C_n ,where n $\geqslant$ 5 and P_5 as a subgraph of C_n, $\lambda(C_n) \geqslant \lambda$ (P_5) = 4.Now we want to show $\lambda(C_n) \leqslant 4$ by defining a labelling on C_n using no label greater than 4. We have 3 cases.

First suppose n $\equiv$ 0(mod 3).Then we can label our vertices (starting at one vertex and proceeding clockwise) 0,2,4,0,2,4... Next suppose n$\equiv$ 1(mod 3).Then we label our vertices 0,2,4,0,2,4...0,2,4,0,3,1,4.
If n$\equiv$ 2 (mod 3), then we can label our vertices 0,2,4,0,2,4...0,2,4,1,3.
This is illustrated in the following figure. In each case, we repeat the labelling 0,2,4 as many times as necessary.This completes the proof.



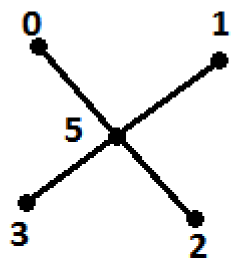
4. STAR GRAPH (S_n)

S_1 is a single vertex, S_2 and S_3 are similar to path with vetices 2 and 3 respectively.
(i) S_4



$\lambda(S_4) = 4$

(ii) S_5



$\lambda(\mathrm{S}_5) = 5$

In general we can say that $\lambda(\mathrm{S}_n)$ = n, where n is the number of vertices.

CHAPTER-2

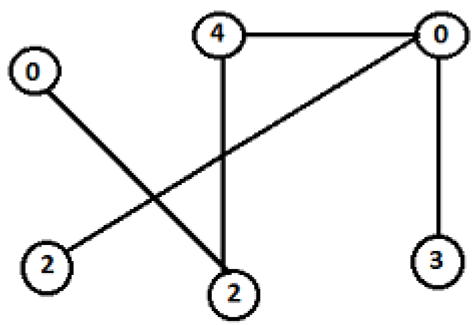
GREEDY ALGORITHM AND BOUNDS

ALGORITHM-1

2.1 GREEDY LABELLING

For a given graph with vertices $v_1,v_2,v_3...v_n$,label vertices in the order $v_1,v_2,v_3...v_n$ by assigning the smallest non-negative integer s such that |s-r| $\geqslant$ 2 for any r assigned to a lower indexed neighbour and |s-r| $\geqslant$ 1 for any t assigned to a lower indexed vertex at a distance 2.

Following figure given an example of a graph with ordered vertices and its greedy labelling.



Now we are able to prove an easy bound.

THEOREM-1

If G is a graph of order n,then λ(G) n + χ (G)-2

Proof:

Suppose that χ(G)=k.Then V(G) can be partitiond into k independent sests v_1,v_2,v_3 ...v_k where | v_1 | = n_i for 1⩽i⩽k.Assign the labels 0,1,2...n_i-1 to the vertices of v_1,for 2≤i≤k assign the labels

$$n_1+n_2+...+n_{i-1}+(i-1),$$
$$n_1+n_2+...+n_{i-1}+i,$$
$$.$$
$$.$$
$$.$$
$$n_1+n_2+...+n_i+(i-2),$$

to the vertices of v_i,since this is an L(2,1) labelling of G, it follow that λ(G)$\leqslant$n+k-2 as desired.

An immediate consequence of this theorem is the following.

COROLLARY-2

If G is a complete k partite graph of order n,where k$\geqslant$ 2 then λ(G)=n+k-2.

Proof:

Let G be a complete k-partite graph with partite sets $v_1,v_2,v_3...v_k$.By above theorem λ(G)$\leqslant$n+k-2.
Let c be an L(2,1) labelling of G with span λ(G) using labels from the set s= $\{0, 1, 2...\lambda(G)\}$ and let a_i be the largest label assigned to a vertex of v_i (1$\leqslant$i$\leqslant$k).Since every two distinct vertices of G are either adjacent or at distance 2, it follows that c must assign distinct labels to all n vertices of G.Furthermore,since every 2 vertices of G belonging to different partite sets are adjacent,it follows that no vertex of G can be labelled a_{i+1} for any (1$\leqslant$i$\leqslant$k). hence there are k-1 labels of s that cannot be assigned to any vertex of G,which implies that the largest label that c can assign to a vertex of G is atleast (n-1)+(k-1) =n +k-2 and so λ(G) $\geqslant$ n+k-2.
Therefore λ(G) = n+k-2.

THEOREM-3

Let G be a graph with maximum degree Δ.Then λ(G)$\leqslant \Delta^2 + 2\Delta$

Proof:

For a given sequence $v_1,v_2,v_3...v_n$,of the vertices of G,we now conduct a greedy algorithm.
A vertex $v \in V(G)$ has atmost Δ neighbours.Each of these neighbour can rule out atmost 3 labels for v (eg: if v is neighbour with a vertex labelled 2, it cannot be labelled 1,2 or 3).
For each neighbour of v, there are atmost Δ-1 vertices a distance two from v(since we don't consider v at distance two from itself). So there are a total of atmost $\Delta(\Delta\text{-}1) = \Delta^2\text{-}\Delta$ vertices a distance two away from v.
Each of these vertices can rule out atmost 1 label for v.Because, if v_j is a vertex at distance two from v and preceeds v in the sequence,then we must avoid assigning v the label given to v_j.
Thus when it comes time to label v, there are atmost $3\Delta+\Delta^2\text{-}\Delta = \Delta^2+2\Delta$ numbers we must avoid.So v can be labelled with some numbers in $\{0,1,2...,\Delta^2+2\Delta\}$.
Therefore, $\lambda(G) \leqslant \Delta^2 + 2\Delta$

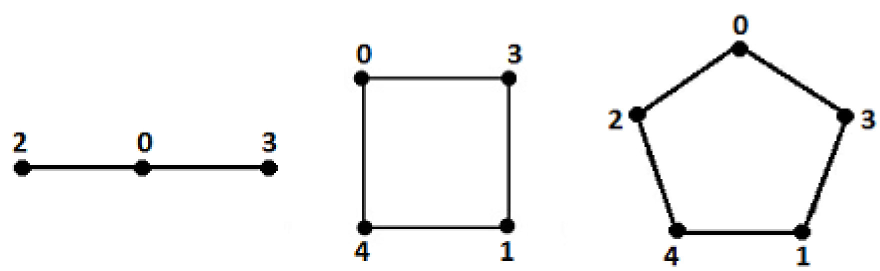
Griggs and Yeh also showed that if a graph G has diameer 2,then the bound $\Delta^2+2\Delta$ for $\lambda(G)$ in the above the above theorem can be improved.

THEOREM-4

If G is a connected graph of diameter 2 with $\Delta(G) = \Delta$ then $\lambda(G) \leqslant \Delta^2$.

Proof:

If $\Delta = 2$, then G is either P_3,C_4 or C_5.The L(2,1) labelling of these three graphs in the following show that $\lambda(G) \leqslant 4$ for each such graph G. Hence we can now assume that $\Delta \geqslant 3$. Suppose that the order of G is n.We consider two cases for Δ , according to whether Δ is large or small in comparison with n.



Case-1 : $\Delta \geqslant \frac{(n-1)}{2}$

Since G is neither a cycle nor a complete graph it follows from Brook's theorem that χ (G) $\leqslant \Delta$.
By theorem-1

$$\lambda(G) \leqslant n+\Delta(G)-2 \leqslant (2\Delta+1)+\Delta-2\ 3\Delta-1 \leqslant \Delta^{2}$$

the final inequality follows because $\Delta \geqslant 3$.

Case-2: $\Delta \leqslant \frac{(n-1)}{2}$

Therefore $\delta\ (\overline{G}) \geqslant \frac{n}{2}$

$\overline{G}$ is hamiltonian and so contains a hamiltonian path P=(v_1,v_2,v_3...v_n)

Define a labelling c on G by c(v_i)=i-1 for $1 \leqslant i \leqslant n$.
Since every two vertices of G with consecutive labels are adjacent in $\overline{G}$,these vertices are not adjacent in G.Thus c is an L(2,1) labelling of G and the c-span is n-1, which implies that λ(G) $\leqslant$ n-1.
Now, for each vertex v of G, atmost Δ vertices are adjacent to v and atmost Δ^{2}- Δ are at distance 2 from v.Since the diameter of G is 2, all vertices of G are within distance 2 of v and so,

$$n \leqslant 1 + \Delta + (\Delta^{2} - \Delta) = \Delta^{2} + 1$$

Therefore, λ(G) $\leqslant$ n-1 $\leqslant \Delta^{2}$.

2.2 L(2,1) LABELLING OF A FINITE PROJECTIVE PLANE.

DEFINITION

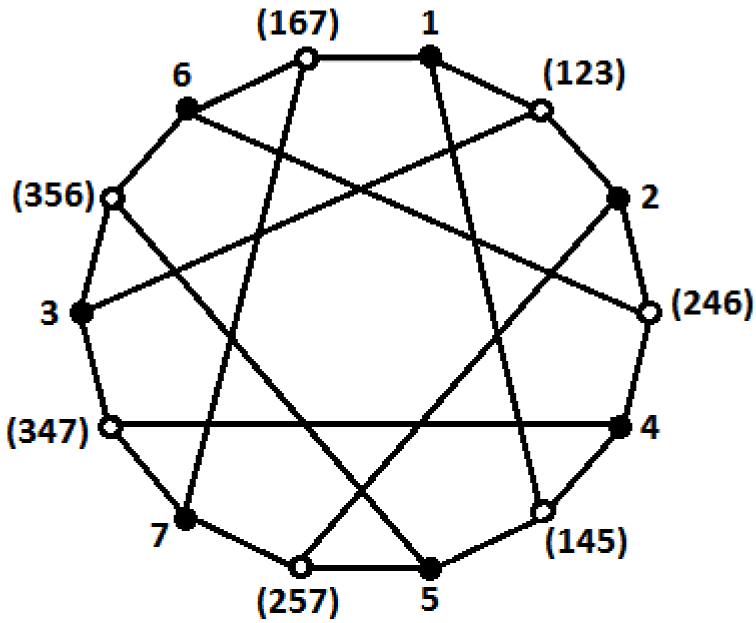
A finite projective plane of order $n \geqslant 2$ is a set of n^2+n+1 objects called points and a set of n^2+n+1 objects called lines such that ech point is incident with n+1 points.
If n is a power of a prime ,then a projective plane of order n exits.
In particular,there is a projective plane of order 2 (containing $2^2+2+1 = 7$ points and 7 lines) and a projective plane of order 3 (containing 13 points and 13 lines).
Griggs and Yeh discribes a class of graphs G with maximum degree Δ for which $\lambda(G) = \Delta^2 - \Delta$.
These are the incidence graphs of finite projective planes.

The incidence graph of a projective plane of order n is a bipartite graph G with partite sets v_1 and v_2 ,where v_1 is the set of points and v_2 is the set of lines and uv is an edge of G if one of u and v is a point and the other is a line incident with this point.

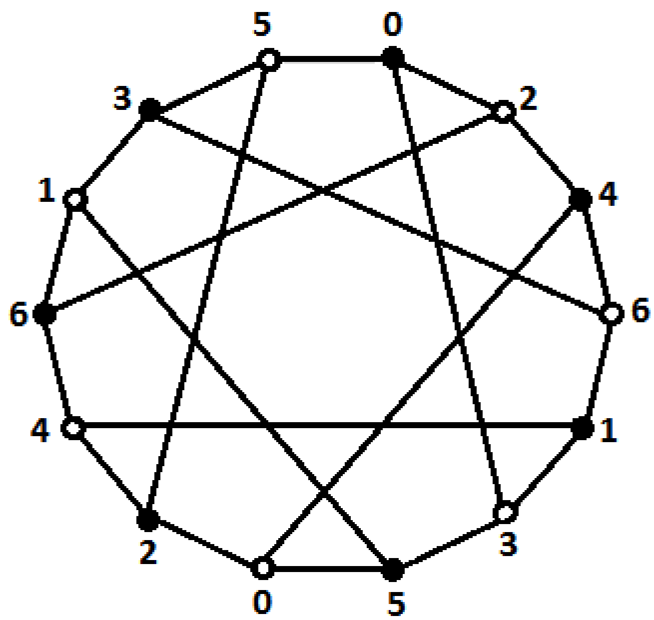
Thus $| v_1 | = | v_2 | = n^2 + n+1$ and so G is an (n+1)-regular bipartite graph of order $2(n^2+n+1)$.

In the simple case, the projective plane of order 2 (also called the Fano plane) is a 3-regular graph of order 14.In this case,the set of points can be denoted by, $v_1 = \{1, 2, 3, 4, 5, 6, 7\}$ and the set of lines by $v_2 = \{(123), (246), (145), (257), (347), (356), (167)\}$.

The incidence graph of this projective plane is shown below.



In the incidence graph G of a projective plane of order n,the distance beween every two vertices of v_i (i=1,2) is 2.so no two vertices of v_1 and v_2 can be assigned the same label in an L(2,1) labelling of G.
This says that $\lambda(G) \geqslant n^2$+n. Because there is an L(2,1) labelling of G using the labels 0,1,2...n^2+n.
Thus we get $\lambda(G)=\Delta^2 - \Delta$.
Therefore the L-span of the incidence graph G of the projective of order 2 is 6.An L(2,1) labelling of this graph using the labels 0,1,2...6 is shown in the figure below.



Griggs and Yeh conjecture:

If G is a graph with $\Delta(G) = \Delta \geqslant 2$ then $\lambda(G) \leqslant \Delta^{2}$

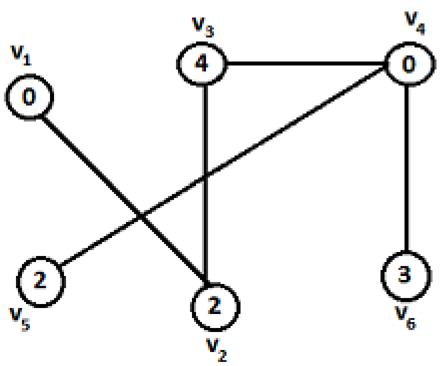
CHAPTER-3

THE CHANG-KUO ALGORITHM

3.1 k-STABLE SET

For any fixed positive integer k,a k-stable set of a graph G is a subset S of V(G) such that every two distinct vertices in S are of distance greater than k.

EXAMPLE:



In this figure $\{v_1,v_4\}$ from a 2-stable set.Since v_1 and v_4 are more than a distance 2 apart.Similarly $\{v_2,v_5\}$ also form a 2-stable set.

NOTE

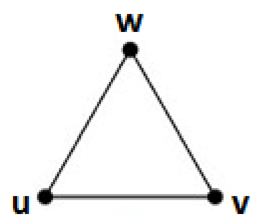
Every vertex in a 2-stable set can be assigned the same label without violating any of our conditions.
The next algorithm uses this idea to give a labelling scheme.

3.2 ALGORITHM

For any graph G, start with all vertices unlabelled.Let $S_{-1} = \phi$
When S_{i-1} is determined and not are vertices in G are labelled,let
$F_i = \{x \in V(G) \; / x$ is unlabelled and $d(x,y) \geqslant 2 \; \forall \; y \in S_{i-1}\}$

Choose a maximal 2-stable subset S_i of F_i.Label all vertices in S_i by i. Increase i by one and continue the above process until all vertices are labelled.

EXAMPLE:



K_3

to see how this works, apply the above algorithm to K_3.
First we have all vertices unlabelled and $S_{-1} = \phi$
Now we determine F_0.
Since all vertices at this point, are unlabelled and it is vacously true that all vertices are atleast a distance 2 from all vertices in the empty set.We have that $F_0=\{u, v, w\}$.
Next,we determine S_0 by choosing a maximal 2-stable subset of $F_0=\{u, v, w\}$.
Let's have $S_0 = \{u\}$
and label u with 0.Now we determine F_1.Since no verex is atleast a distance 2 from u,we have,

$$F_1=\phi$$
$$\text{so } S_1=\phi$$

Now we determine F_2.
Since both of our remaining unlabelled vertices are atleast a distance 2 from all vertices in the empty set,we have that,

$$F_2=\{v, w\}$$

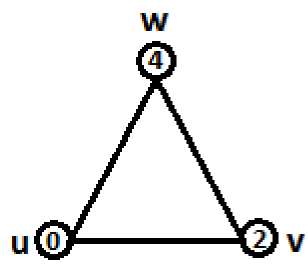
Next ,we determine S_2 by choosing a maximal 2-stable subset of $F_2 = \{v, w\}$ .Let's have,

$$S_2=\{v\}$$

and labe v with 2.
Now we determine F_3.
Since no vertex is atleast a distance 2 from v,we get,

$$F_3 = \phi$$
$$\text{so } S_3=\phi$$

Now we determine F_4. Since our remaining unlabelled vertex is atleast a distance 2 from all the vertices in the empty set we have thatt $F_4=\{w\}$. Which means $S_4=\{w\}$.
and we label w with 4.The finished labelling is shown below.



THEOREM

Let G be a graph with maximum degree Δ.Then $\lambda(G) \leqslant \Delta^2+\Delta$

Proof:

Let G be a graph with maximum degree Δ.Perform Chang-kuo algorithm on G.Let k be the maximum label used and let x be a vertex whose label is k.Let,
$I_1 = \{i|\ 0 \leqslant i \leqslant k\text{-}1 \text{ and } d(x,y) = 1 \text{ for some } y \in S_i\}$,
$I_2 = \{i|\ 0 \leqslant i \leqslant k\text{-}1 \text{ and } d(x,y) \leqslant 2 \text{ for some } y \in S_i\}$,
$I_3 = \{i|\ 0 \leqslant i \leqslant k\text{-}1 \text{ and } d(x,y) \geqslant 3 \text{ for all } y \in S_i\}$.
Then we have,

$$| I_2 | + | I_3 | = k.$$

Since the total number of vertices y with $1 \leqslant d(x,y) \leqslant 2$ is atmost $\Delta + \Delta(\Delta-1) = \Delta^2$,we have

$$| I_2 | \leqslant \Delta^2$$

Also,there are atmost Δ vertices adjacent to x so

$$|I_1| \leqslant \Delta$$

Now for any $i \in I_3, x \notin F_i$.Otherwise, $S_i \cup \{x\}$ is a 2-stable subset of F_i,which contradicts the choice of a maimal S_i.This means d(x,y) = 1 for some vertex $y \in S_{i-1}$.So, $i-1 \in I_1$.Thus,

$$| I_3 | \leqslant | I_1 |.$$

Therefore, combining (1),(2),(3) and (4) gives

$$\lambda(G) \leqslant k = | I_2 | + | I_3 | \leqslant | I_2 | + | I_1 | \leqslant \Delta^2 + \Delta.$$

3.3 DIFFERENCE BETWEEN GREEDY LABELLING AND THE CHANG-KUO ALGORITHM

The greedy labelling algorithm goes through each vertex and assigns it the smallest possible label,whereas the chang kuo algorithm goes through each label and assigns it to a maximal set of posssible vertices.

CONCLUSION

The main goal of l(2,1) labelling problems is to find a labelling with minimum span or distance between the highest and lowest labels used. Here we are mainly focussed on the upper bounds on the L(2,1) labelling numbers of graphs with maximum degree Δ

Griggs and Yeh used greedy algorithm to establish an upper bound for λ in terms of Δ.
Later the upper bound theorem was modified by Chang and Kuo .
There is no proof for the Griggs and Yeh conjecture in general.

Also we can observe that L(0,1) labelling which is a particular case of L(p,q) labelling is not a proper coloring.So L(p,q) labelling is a proper coloring if $p > 0$.

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