STUDY ON HYPERGRAPHS

A Dissertation submitted in partial fulfillment of the

Requirement for the award of

DEGREE OF MASTER OF SCIENCE

IN MATHEMATICS

By

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CERTIFICATE

This is to certify that the dissertation entitled" STUDY ON HYPERGRAPHS" is a bonafide record of the work done by SHEMINA M.M under my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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DECLARATION

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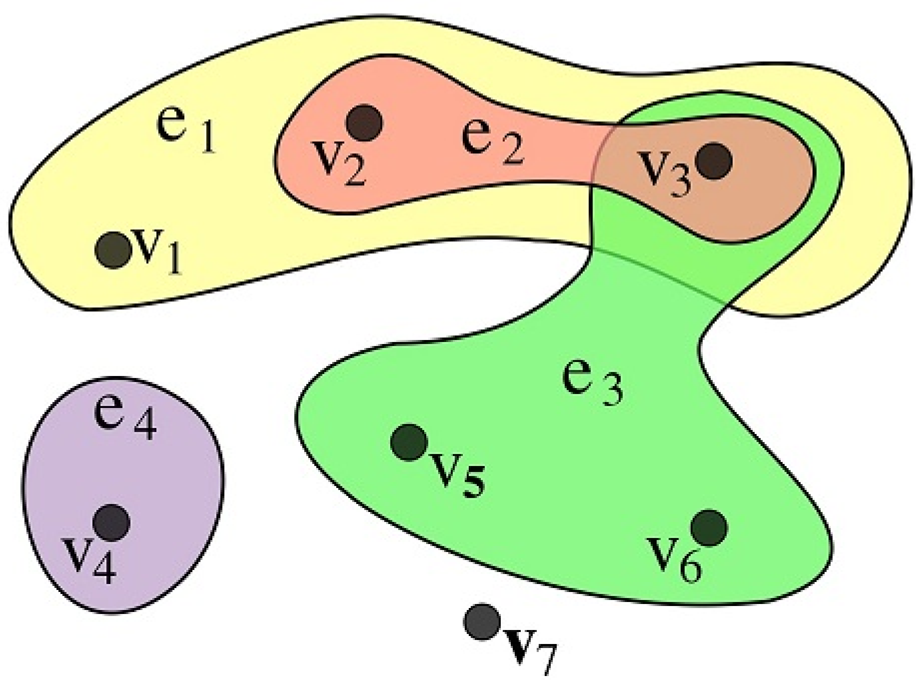
CONTENTS

INTRODUCTION

In mathematics, a hypergraph is a generalization of a graph in which an edge can join any number of vertices. Formally, a hypergraph H is a pair H = (X,E) where X is a set of elements called nodes or vertices, and E is a set of non-empty subsets of X called hyperedges. Therefore, E is the subset of P(X)/ϕ, where P(X) is the power set of X.

While graph edges are pair of nodes, hyperedges are arbitrary sets of nodes, and can therefore contain an arbitrary number of nodes.



An example of a hypergraph, with X = $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and E = $\{e_1, e_2, e_3, e_4\} = \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\}$.

It is often desirable to study hypergraphs where all hyperedges have the same cardinality; a k-uniform hypergraph is a hypergraph such that all its hyperedges have size k. In other words, one such hypergraph is a collection of sets, each such set a hyperedge connecting k nodes. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. A hypergraph is also called a system or a family of sets drawn from the universal set X.

Hypergraphs have many other names. In computational geometry, a hypergraph may sometimes be called a range space and then hyperedges are called ranges. In some literatures edges are referred to as hyperlinks and connectors.

PRILIMINARIES

GRAPH : It is an ordered triple, (V(G), E(G), ψ_G) consisting of non empty set V(G), a disjoint set E(G) of edge set and an incidence function ψ_G that associates with each edge of G and an ordered pair of vertices (same or distinct) of G.

LOOP: An edge with identical ends.

TRIVIAL GRAPH: A graph with one vertex and has no edge.

ORDER AND SIZE: The number of vertices of graph G is called the order of graph, denoted by V(G). Similarly, the number of edges of G is called size of graph, denoted by E(G).

BI-PARTIATE GRAPH: A graph is bi-partiate if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge has one end in X and other end in Y. Such a partition (X,Y) is called a bipartition of the graph.

COMPLETE BI-PARTIATE: It is a simple bi-partiate graph with the bi-partition (X, Y) in which each vertex of X is joined to each vertex of Y.

SUBGRAPH: A graph H is a subgraph of graph G if V(H)⊆V(G) and E(H)⊆E(G) .

INDUCED SUBGRAPH: Let V' be a non empty subset of V, the subgraph of G whose vertex set is V' and the edge set is the set of those edges in G whose both vertices are in V' is called a subgraph induced by V'.

WALK: A walk in a graph G is an alternating sequence W = $v_0 e_1 v_2 ... e_k v_k$ whose terms are alternating vertices and edges beginning and ending with vertices in which v_{i-1}, v_i are ends of e_i. Here v_0 is the origin and v_k is the terminus.

- A $v_0 - v_k$ walk is closed if $v_0 = v_k$, otherwise it is open.
- A walk is called a trail if all the edges appearing in the walk are distinct.
- A walk is called a path if all the vertices are distinct.

CONNECTEDNESS: Two vertices u and v of G are said to be connected if there is a u-v path in G. Every vertex u of the graph G is connected to itself.

CONNECTED GRAPH: A graph G connected if every two vertices of G are connected. i.e; There exist a path joining every two vertices of G. A graph that is not connected is called a disconnected graph.

EDGE GRAPH OR LINE GRAPH: The line graph of a loopless graph G is the graph with vertex set E(G) in which two vertices are joined if and only if they are adjacent in G. Denoted by L(G).

VERTEX DEGREE: The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges.

EULER'S THEOREM OR FIRST THEOREM OF GRAPH THEORY

$\Sigma_{v \in V}$ d(v) $= 2\varepsilon$
i.e; The sum of degrees of vertices of a graph is equal to twice the number of edges.

ACYCLIC GRAPHS: A graph without any cycles.

TREE: Connected acyclic graph.

CHAPTER 1

BASIC CONCEPTS

1.1 BASIC DEFINITIONS

A hypergraph H is a pair H=(V,E=e_i ; i ∈ I) where V is the set of elements called vertices or nodes and E is the nonempty subsets of V called hyper-edges or edges. Sometimes V is denoted V(H) and E by E(H).The order of a hypergraph is the cardinality of V and its size is the cardinality of E.

A hypergraph with a single vertex is called trivial, and a hypergraph with no edges is called empty.A hypergraph called simple if no edge is contained in another.

DEFINITION

Let H =(V,E) be a hypergraph. If v,w ∈ V are distinct vertices and there exists e ∈ E such that v,w ∈ e,then v and w are said to be adjacent in H. Similarly, if e,f ∈ E are distinct edges and v ∈ V is such that v ∈ e $\bigcap$ f, then e and f are said to be adjacent in H.

Each ordered pair (v,e) such that v ∈ V, e ∈ E,and v ∈ e is called a flag of H; the set of flags is denoted by F(H).If (v,e) is a flag of H,then we say that vertex v is incident with edge e.

The degree of a vertex v ∈ V (denoted by deg(v)) is the number of edges e such that v ∈ e. A vertex of degree zero is called isolated vertex , and a vertex of degree one is called pendant vertex.A hypergraph H is regular of degree r(or r- regular) if every vertex of H has degree r .

Th maximum cardinality |e| of any edge e ∈ E is called the rank of H. Similarly, the minimum cardinality |e| of any edge e ∈ E is called corank of H .A hypergraph H is uniform rank (r-uniform) if |e| = r for all e ∈ E. An edge e ∈ E is called a singleton edge if |e| = 1, and empty if |e| = 0.
A hypergraph is called simple if no edge is contained in another.

DEFINITION

Let H=(V,E) be a hypergraph .

1. A hypergraph H' = (V',E') is called a subhypergraph of H if V' $\subseteq$ V and either E'=ϕ or $\{e \cap V' : e \in E, e \cap V' \neq \phi\}$.

2. A subhypergraph H'=(V',E') of H with E'= $\{e \cap V' : e \in E, e \cap V' \neq \phi\}$ is said to be induced by V'.

3. If |V| $\geq$ 2 and v $\in$ V, then H|v will denote the subhypergraph of H induced by V - $\{v\}$, also called a vertex-deleted subhypergraph of H.

4. A hypergraph H' =(V', E') is called a hypersubgraph of H if V'$\subseteq$ V and E'$\subseteq$ E.

5. A hypergraph H'=(V', E') of H is said to be induced by V', denoted by H$[V']$, if E' = $\{e \in E : e \neq \phi\}$.

6. A hypergraph H' = (V',E') of H is said to be induced by E', denoted by H$[E']$,if V'=$\cup_{e \in E}$e.

7.For E' $\subseteq$ E, we write shortly H -E' and H-e for the hypergraphs H (V,E-E') and $(V, E - e)$, respectively. The hypersubgraph H-e may also be called an edge-deleted hypersubgraph.

8. A hypersubgraph H'=(V',E') of H is called spanning if V' = V.

REMARK

1. The vertex-deleted subhypergraph H|v is obtained by removing vertex v from V and from all edges of H,and then discarding the empty edges.

2. It is easy to see that every hypersubgraph of H=(V,E) is also a subhypergraph of H, but not conversely. However,not every hypersubgraph of H induced by V' $\subseteq$ V is a subhypergraph of H induced by V'.

3. Edge-deletion as defined above is called weak edge deletion, and weak vertex deletion is defined as our vertex deletion except that empty edges are not discarded.To stongly delete a vertex v from a hypergraph H=(V,E), we remove a vertex v from V and remove all edges containing v from E. To strongly delete an edge from H, we remove edge e from E, as well as all vertices contained in e from both V and all edges incident with them.

1.2 CONNECTION IN HYPERGRAPHS

WALKS, TRAILS, PATHS, CYCLES

DEFINITION

Let H=(V,E) be a hypergraph, let u,v $\in$ V and let k $\geq$0 be an integer.
A (u,v)-walk of length k in H is a sequence $v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$ of vertices and edges such that v_0,v_1,...,v_{k-1},v_k $\in$ V, e_1, e_2,..., e_k $\in$ E, v_0 = u , v_k = v, and for all i=1,2,...,k, the vertices v_{i-1} and v_i are adjacent in H.

Let W= $v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$ is a walk in H, then the vertices v_0 and v_k are called end points of W, and v_1,...,v_{k-1} are the internal vertices of W. Furthermore, vertices v_0,v_1,...,v_k are called the anchors of W, and any vertices u $\in$ e_i, for some i $\in$ $\{1, 2, ..., k\}$, that is not an anchor of W.

DEFINITION

Let W=$v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$ be a walk in a hypergraph H=(V,E).
1. If the anchor flags (v_0, e_1),(v_1, e_1),(v_1, e_2),...,(v_{k-1}, e_k),(v_k, e_k) are pairwise distinct, then W is called a trail.

2. If the edges e_1,....,e_k are pairwise distinct, then W is called a strict trail.

3. If the anchor flags (v_0, e_1),(v_1, e_1),(v_1, e_2),...,(v_{k-1}, e_k),(v_k, e_k) and the vertices v_0,v_1,...,v_k are pairwise distinct (but the edges need not be), then W is called a pseudo path.

4. If both the vertices v_0,v_1,...,v_k and the edges e_1,....,e_k are pairwise distinct, then W is called a path.

DEFINITION

Let W = $v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$ be a walk in a hypergraph H=(V,E). If k $\geq$ 2 and $v_0 = v_k$, then W is called a closed Walk. Moreover:

1. If W is a trail (strict trail) , then it is called a closed trail (closed strict trail) respectively.

2.If W is a closed trail and vertices v_0,v_1,...,v_{k-1} are pairwise distinct (but edges need not be), then W is called pseudo cycle.

3. If the vertices v_0,v_1,...,v_{k-1} and the edges e_1,...,e_k are pairwise distinct, then W is called a cycle.

LEMMA

Let W be a walk in a hypergraph H. Then:

1. If W is a trail, then no two consecutive edges in W are the same (including the last and the first edges if W is a closed trail).

2. If W is a (closed) strict trail, then it is a (closed) trail.

3. If W is a pseudo path (pseudo cycle), then it is a trail (closed trail),but not necessarily a strict trail (closed strict trail).

4. If W is a path (cycle), then it is both a pseudo path (pseudo cycle) and a strict trail (closed strict trail).

DEFINITION

Let W=$v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$ and W' = $v_ke_{k+1}v_{k+1}...$ e_lv_l, for 0$\leq$ k $\leq$ l, be two walks in a hypergraph H = (V,E). The concatenation of W and W' is the walk WW' = $v_0e_1v_1e_2v_2...v_{k-1}e_kv_ke_{k+1}v_{k+1}...$ e_lv_l.

1.3 EXAMPLES OF HYPERGRAPH

EXAMPLE 1

Let M be a mathematics meeting with k ≥ 0 sessions, S_1 , S_2 ,..., S_k. Let V be the set of people at this meeting. Assume that each session is attended by at least one person.

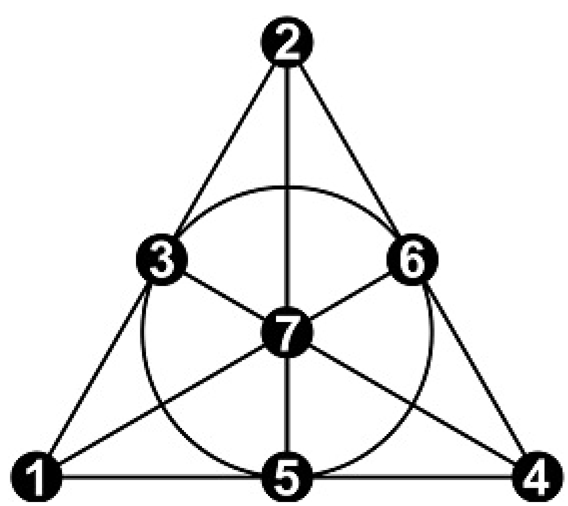
We can build a hypergraph as following way:

- The set of vertices is the set of people attended the meeting.
- The family of hyperedges (e_i) ; i $\in \{i = 1, 2, ..., k\}$ is the subset of the people who attend the meeting S_i ; i $\in \{i = 1, 2, ..., k\}$.

EXAMPLE 2

FANO PLANE

The fano plane is a finite projective plane which has smallest number of points and lines, 7 points with 3 points on every line and 3 lines through every point.



To a fano plane we can associate a hypergraph called fano hypergraph:

- The set of vertices is V = $\{0, 1, 2, 3, 4, 5, 6\}$;
- The set of hyperedges is E =$\{013, 045, 026, 124, 346, 235, 156\}$.

The rank is equal to corank which is equal to 3.
Hence fano hypergraph is 3-uniform.

EXAMPLE 3

STEINER SYSTEMS

Let t;k;n be integers which satisfies: $2 \leq$ t $\leq$ k $<$ n . A steiner system is denoted by s(t;k;n) is a k-uniform hypergraph H = (V,E) with n vertices such that for each subset T $\subseteq$ V with t elements there is exactly one hyperedge e $\in$ E Satisfying T $\in$ e .

1.4 CONNECTED HYPERGRAPHS

Let H = (V,E) be a hypergraph.Vertices u,v $\in$ V are said to be connected in H if there exist a (u,v)-walk in H. The hypergraph H is said to be connected in H.

CONNECTED COMPONENT

Let H = (V,E) be a hypergraph and V' $\subseteq$ V be an equivalent class with respect to vertex connection. The hypergraph of H induced by V' is called a connected component of H. We denote the number of connected components of H by ω (H).

CUT EDGES AND CUT VERTICES

DEFINITION

A cut edge in a hypergraph H = (V,E) is an edge e ∈ E such that
$\omega(H-e) > \omega(H)$.

LEMMA

Let e be a cut edge in a hypergraph H = (V,E).
Then $\omega(H) < \omega(H-e) \leq \omega(H)$ + | e | - 1.

PROOF

The inequality on the left follows straight from the definition of cut edge.
To see the inequality on the right , first observe that e is not empty. Let H_1,H_2,...,H_k be connected components of H-e whose vertex set intersect e. Since e has at least one vertex in common with each $V(H_1)$, we have $\mid e \mid \geq$ k.
Hence $\omega(H-e) = \omega(H)$ + k -1 $\leq \omega(H)$ + | e | - 1.

DEFINITION

A cut edge of a hypergraph H is called strong if $\omega(H-e) = \omega(H)$ + | e |- 1, weak otherwise.

THEOREM

Let e be an edge in a connected hypergraph H = (V,E). The following are equivalent:
1. e is a strong cut edge, that is, $\omega(H-e) = \mid e \mid$.
2. e contains exactly one vertex from each connected component of H-e.
3. e lies in no cycle of H.

PROOF

(1) ⇒ (2): Let e be a strong cut edge of H. Since H is connected, the edge e must have at least one vertex in each connected component of H-e. Since there are $\mid e \mid$ connected components of H-e, the edge e must have exactly one vertex in each of them.

(2) $\Rightarrow$ (1): Assume e contain exactly one vertex from each connected component of H - e. Then clearly $\omega(H-e) = \mid e \mid$.

(2) $\Rightarrow$ (3): Assume e contains exactly one vertex from each connected component of H-e, and suppose e lies in a cycle C = $v_0 e_1 v_1 e_2 ... v_{k-1} e v_0$ of H. Then $v_0 e_1 v_1 e_2 ... v_{k-1}$ is a path in H-e, and v_0 and v_{k-1} are two vertices of e in the same connected component of H-e, a contradiction. Hence e lies in no cycle of H.

(3) $\Rightarrow$ (2): Assume e lies in no cycle of H. Since H is connected, the edge e must contain at least one vertex from each connected component of H-e. Suppose e contains two vertices u and v in the same connected component H' of H-e. Then H' contains a (u,v)-path P, and P is a cycle in H that contains e, a contradiction. Hence e possesses exactly one vertex from each connected component of H-e.

Corollary

Let e be an edge in a hypergraph H = (V,E). The following are equivalent:

1. e is a strong cut edge, that is , $\omega(H-e) = \omega(H) + \mid e \mid$ - 1.

2. e contains exactly one vertex from each connected component of H-e that it intersects.

3. e lies in no cycle of H.

DEFINITION

A cut vertex in a hypergraph H = (V,E) with $\mid V \mid \geq 2$ is a vertex v $\in$ V such that $\omega(H \mid v) > \omega(H)$.

1.5 MATRIX ASSOCIATED WITH HYPERGRAPHS

INCIDENCE MATRIX

Let H = (V,E) be a hypergraph, V = (v_1,v_2,...,v_n) and E = (e_1,e_2,...,e_m) with $\bigcup_{i \in I} e_i$ = V (without isolated vertex).
Then H has an n×m incidence matrix A = (a_{ij});
a_{ij} = 1 if $v_i \in e_i$ and 0 otherwise.
This matrix may also write as m×n matrix.

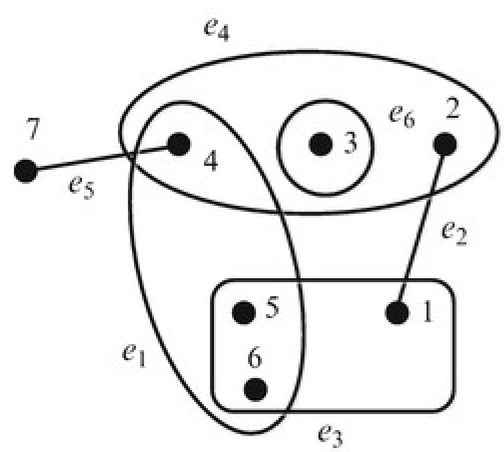


Figure 1.2 A hypergraph

Representation of above hypergraph by incidence matrix:

	e1	e2	e3	e4	e5	e6
1	0	1	1	0	0	0
2	0	1	0	1	0	0
3	0	0	0	1	0	1
4	1	0	0	1	1	0
5	1	0	1	0	0	0
6	1	0	1	0	0	0
7	0	0	0	0	1	0

CHAPTER 2
FIRST PROPERTIES

In the first chapter we saw that hypergraphs generalize standard graphs by defining edges between multiple vertices instead of 2 vertices. Hence some properties must be generalization of graph properties. In this chapter , we introduce some basic properties of hypergraphs which will be used throughout.

GRAPHS

A multiple graph G = (V,E) is a hypergraph such that the rank is atmost 2. The hyperedges are called edges. If the hypergraph is simple without loop, it is a graph.Consequently other definitions of hypergraph hold for graphs.

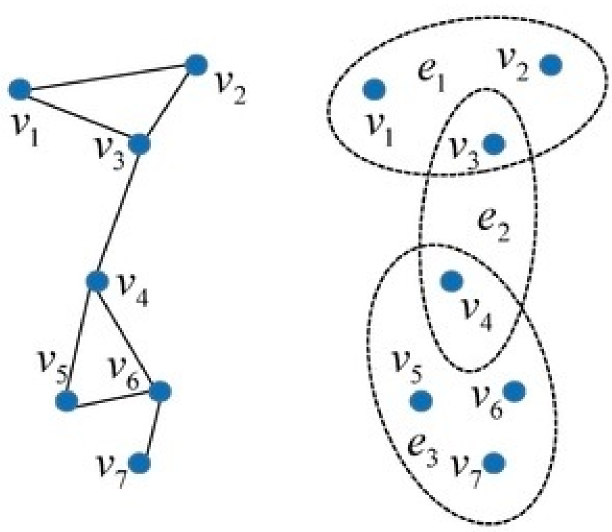


Figure 2.1 (a) represents a graph , (b) represents a hypergraph.

2.1 LINE GRAPH OF A HYPERGRAPH

Let H = $(V, E = (e_i); i \in I)$ be a hypergraph such that E $\neq \phi$. The line graph (or representative graph) of H is the hypergraph H' = (V',E') whose vertex set is the hyperedges of the hypergraph with two hyperedges adjacent when they have a non empty intersection and is denoed by L(H).
ie;
1. V' = E when H is without repeated hyperedges.

2. $e_i, e_j \in$ E' ; i $\neq$ j if and only if $e_i \cap e_j \neq \phi$.

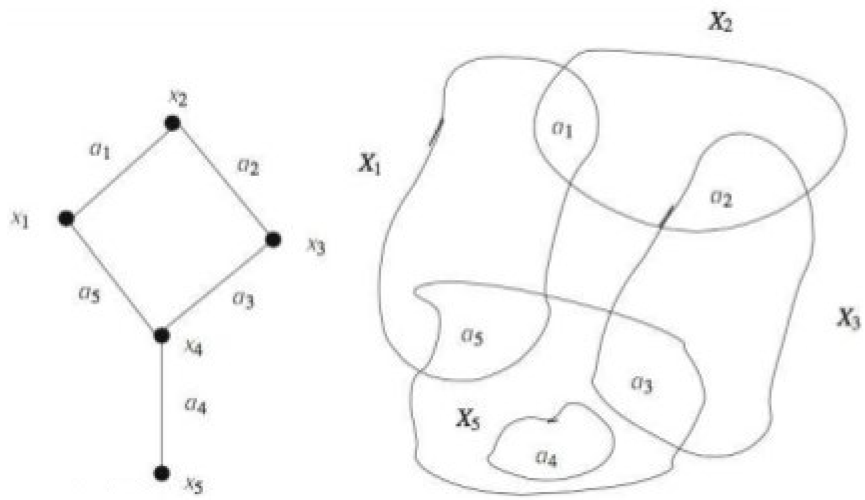


Figure 2.2 Above figure shows a hypergraph H = (V,E) with vertices V = $\{a_1, a_2, a_3, a_4, a_5\}$ and edges E = $\{x_1, x_2, x_3, x_4, x_5\}$ and its line graph where vertices of L(H) are the black dots and its edges are curves between these dots.

LEMMA

The hypergraph H is connected if and only if L(H) is connected.

PREPOSITION

Any non trivial graph G is a line graph of linear hypergraph.

PROOF

Let G = (V,E) be a graph with V = $\{x_1, x_2, ..., x_n\}$.
Without loss of generality, we suppose that G is connected(otherwise we treat the connected component one by one).
We construct a hypergraph H = (W,X) in the following ways;

• The set of vertices is the set of edges of G, ie; W = E.It is possible since G is connected.
• The collection of hyperedges X is the family of X_i; X_i is the set of edges of G having x_i as incidence vertex.
So e can write;
H = $(E, X = (X_1, X_2, ..., X_n))$ with $X_i = \{e \in E; x_i \in e\}$ where i 1,2,...,n.

Notice that if G has only one edge then V = $\{x_1, x_2\}$ and $X_1 = X_2$.
It is the only case where H has repeated hyperedge.

If $\mid E \mid > 1$, if i $\neq$ j and $X_i \cap X_j \neq \phi$; there is exactly one, (since G is simple graph) e $\in X_i \cap X_j$ with e = $\{x_i, x_j\}$. Thus it is clear that G is the line graph of H.
Figure 2.2 illustrates the above preposition.

PREPOSITION

Let H = (V,E) be a hypergraph, we have ;

$\Sigma_{x \in V}$ d(x) = $\Sigma_{e \in E}$ d(e).

PROOF

Let $I_G(H)$ be the incidence graph of a hypergraph H. We sum in the two parts of $I_G(H)$. Since the sum of the degrees in these two parts are equal, we obtain the result.

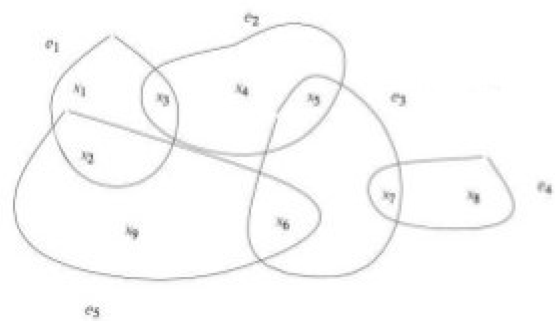


Figure 2.3 A hypergraph which has 9 vertices and 5 hyperedges

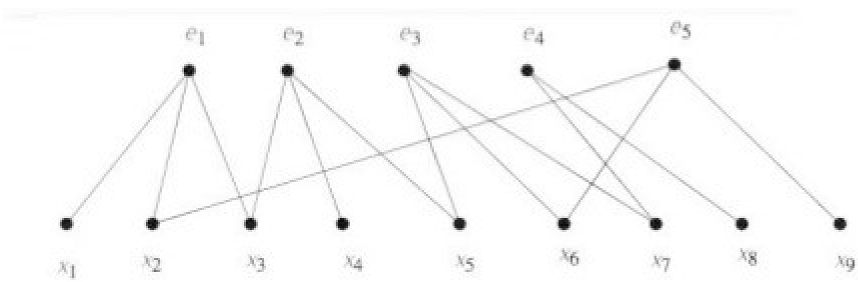


Figure 2.4 The incidence graph associated with above hypergraph

2.2 DUAL HYPERGRAPH

The dual of a hypergraph H = (V, E) ; E = $\{E_1, E_2, ..., E_m\}$ on V is a hypergraph $H^* = (E, X_1, X_2, ..., X_n)$ whose vertices $e_1, e_2, ..., e_m$ corresponds to the edge of H, and with edges $X_i = \{e_j \mid x_i \in E_j\}$; i $\in \{1, 2, ..., n\}$.
Clearly H^* satisfies the conditions of hypergraph.

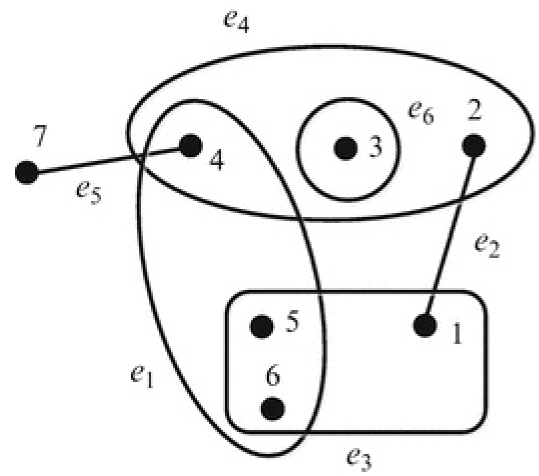


Figure 2.5 A hypergraph

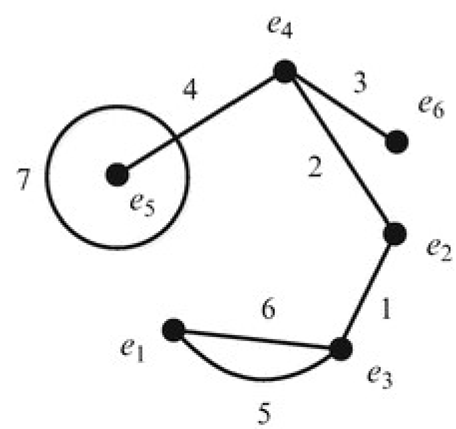


Figure 2.6 Dual of above hypergraph

PREPOSITON

The dual H^* of a linear hypergraph without isolated vertex is also linear.

PROOF

Let H be a linear hypergraph.
Assume that H^* is not linear.

ie; There are two distinct hyperedges X_i , X_j of H^* which intersect with atleast two vertices say e_1 and e_2.By definition of duality,
$X_i = \{e_1, e_2 \mid x_i \in E_1, E_2\}$
$X_j = \{e_1, e_2 \mid x_j \in E_1, E_2\}$
ie; x_i,$x_j \in E_1$ and x_i,$x_j \in E_2$.
x_i,$x_j \in E_1 \cap E_2$
This is a contradiction to the fact that H is linear.
Therefore our assumption is wrong. ie; H^* is linear.

2.3 INTERSECTING FAMILIES, HELLY PROPERTY

INTERSECTING FAMILIES

Let H = $(V, E = (e_i); i \in I)$ be a hypergraph. A subfamily of hyperedges (e_j), j∈J ; J⊆I is an intersecting family if every pair of hyperedges has non empty intersection. The maximum $\mid J \mid$ (of an intersecting family of H) is denoted by Δ_0(H).

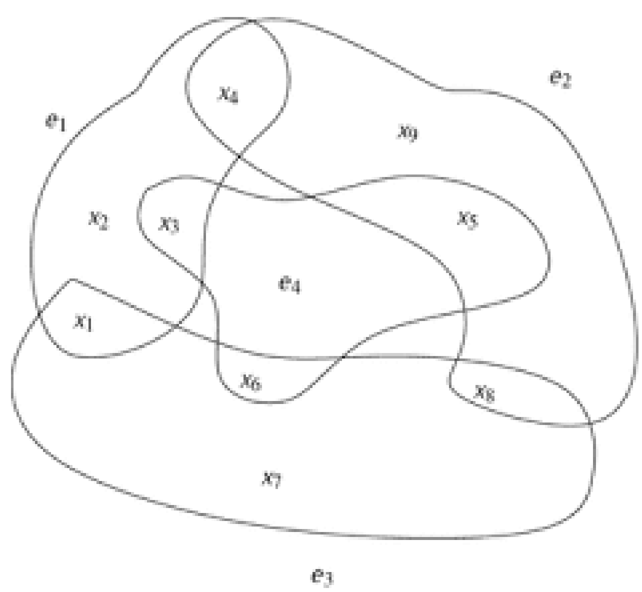


Figure 2.7 An intersecting family

A Star H(x) centered in x is the family of hyperedges (e_j), j∈J containing x.

The maximum $\mid J \mid$ is denoted by Δ(H). Since a star is an intersecting family , obviously we have $\Delta_0(H) \geq \Delta(H)$.
An intersecting family with 3 hyperedges e_1,e_2,e_3 and $e_1 \cap e_2 \cap e_3 = \phi$ is called a triangle.

HELLY PROPERTY

The helly property plays a very important role in the theory of hypergraphs as the most important hypergraphs have this property.A hypergraph has the helly property if each intersecting family has a non empty intersection. It is obvious that if a hypergraph contains a triangle, then it will not have the helly property. A hypergraph having the helly property is called a helly hypergraph.
A hypergraph has the strong helly property if each partial induced subhypergraph has the property.

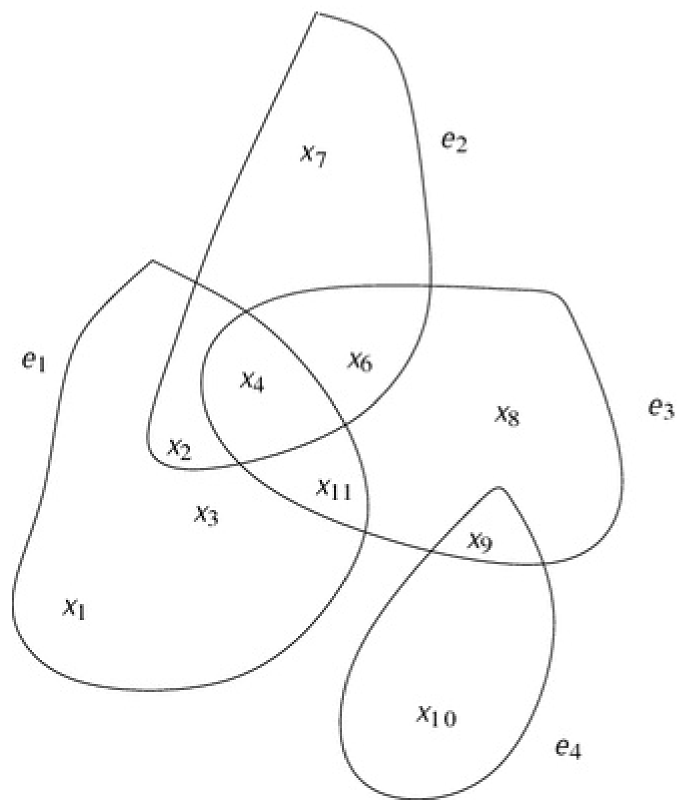


Figure 2.8 Hypergraph above has helly property but not strong helly property.

In the above figure (fig 2.8), the hypergraph has not strong helly property because the induced subhypergraph on Y = V | $\{x_4\}$ contains the triangle: $e_1^{'} = e_1 \cap Y$, $e_2^{'} = e_2 \cap Y$ and $e_3^{'} = e_3 \cap Y$.

THEOREM

Let H be a hypergraph. Any partial induced subhypergraph of H has the Helly property if and only if for any three vertices x,y,z and any three hyperedges e_{xy},e_{yz},e_{xz} of H, where x $\in e_{xy} \cap e_{xz}$, y $\in e_{xy} \cap e_{yz}$, z $\in e_{xz} \cap e_{yz}$ there exist v $\in \{x, y, z\}$ such that v $\in e_{xy} \cap e_{xz} \cap e_{yz}$.

PROOF

Assume that any partial induced subhypergraph of H has the Helly property. Then for any three hyperedges e_{xy},e_{yz},e_{xz} of H, where x $\in e_{xy} \cap e_{xz}$, y $\in e_{xy} \cap e_{yz}$, z $\in e_{xz} \cap e_{yz}$.
Now just take the partial subhypergraph H(Y) induced by the set Y = $\{x, y, z\}$ to see that there is a vertex v $\in \{x, y, z\}$ such that v $\in e_{xy} \cap e_{xz} \cap e_{yz}$ (since it has helly property).

We prove the converse by the method of induction on l, the maximal size of an intersecting family of an induced subhypergraph of H.

Clearly, the assertion is true for l = 3.

Assume that for i = 3, 4,..., l any partial induced subhypergraph of H with intersecting of an induced subhypergraph of H with intersecting families of atmost l hyperedges has helly property.

Now we have to prove that the assertion is true for l+1.
Let $e_1, e_2, ..., l + 1$ be an arbitrary intersecting family of hyperedges of H.
By induction,
$\exists\ x \in \cap_{i\neq 1} e_i$, $\exists\ y \in \cap_{i\neq 2} e_i$, $\exists\ z \in \cap_{i\neq 3} e_i$.
As $\{e_1, e_2, e_3\}$ is an intersecting family, there is a vertex say 'v' which belongs to $\{x, y, z\}$, which is in the intersection $\{e_1 \cap e_2 \cap e_3\}$.
Hence v $\in \cap_i e_i$. Thus the assertion is true for l+1.

2.4 SUBTREE HYPERGRAPHS

Let H = (V,E) be a hypergraph .This hypergraph is called a subtree hypergraph if there is a tree T with vertex set V such that each hyperedge e ∈ E induces a subtree in T.
Conversely, let T=(V,E) be a tree, that is a connected graph without cycle. We can build a hypergraph H in the following way.
• The set of vertices of H is the set of vertices of T;
• The set of hyperedges is the family E= $(e_i; i \in \{1, 2, ..., k\})$ of subset V that induces subgraph T($V(e_i)$) is a subtree of T, (subgraph which is a tree).
For the same tree we may have several hypergraphs generated by above method.

PREPOSITION

Let T=(V,E) be a tree and H be a subtree hypergraph associated with T, H has the helly property.

PROOF

In a tree T, there is exactly one path denoted by $P_a[x, y]$ between two vertices x,y. otherwise T would contain a cycle.
Let u,v,w be three vertices of H. The paths $P_a[u, v]$, $P_a[v, w]$ and $P_a[w, u]$ have one common vertex, otherwise would contain a cycle. Consequently, any family of hyperedges for which every hyperedges contain atleast two of these vertices u,v,w has nonempty intersection.
i.e; Subtree hypergraph associated with a tree T has helly property.

2.5 STABLE OR INDEPENDENT, TRANSVERSAL SET AND MATCHING

Let H=(V,E) be a hypergraph without isolated vertex.
A set A ⊆ V is a stable or independent if no hyperedge is contained in A.
The stability number $\alpha(H)$ is the maximum cardinality of a stable.

A set B ⊆ V is a transversal if it meets every hyperedge. i.e, for every e

$\in$ E, B $\cap$ V(e) $\neq \phi$.

A matching is a set of pairwise disjoint hyperedges of H. The matching number $\nu(H)$ of H is maximum cardinality of a matching.

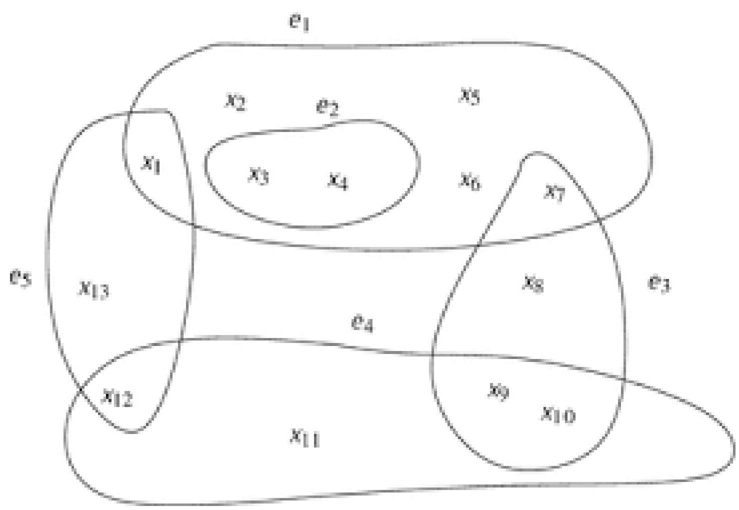


Figure 2.9 The set $\{x_1; x_3; x_5; x_9; x_{11}; x_{13}\}$ is a stable of the hypergraph above but it is not a strong stable. The set $\{x_3; x_8; x_{11}; x_{13}\}$ is a transversal.

EXAMPLES

1.The problem of hiring a set of engineers at the factory is an example of minimum transversal set problem.
Let us suppose that engineers apply for positions with the lists of proficiency they may have, the factory management then tries to hire the least possible number of engineers so that each proficiency that the factory needs is covered by atleast one engineer.
We construct a hypergraph with a vertex for each engineer and a hyperedge for each proficiency. Then a minimum transversal set represents the minimum group of engineers that need to be hired to cover all proficiencies at this factory.

2. The problem of scheduling the presentations in a conference is an

example of the maximum independent set problem.
Let us suppose that people are going to present there works, where each work may have more than one author and each person may have more than one work.
The goal is to attain as many presentations as possible to the same time slot. We constuct the hypergraph with a vertex for each vertex for each person, it is the set of works that he or she presents. Then a maximum strong independent set represents the maximum number of presentations that can be given at the same time.

CHAPTER 3
APPLICATIONS

3.1 CHEMICAL HYPERGRAPH THEORY

The graph theory is very useful in chemistry. The representation of molecular structures by graph is widely used in computational chemistry. But the main drawback of the graph theory is the lack of convinent tools to represent organo metallic compounds, benzenoid system and so on.

A hypergraph H = (V,E) is a molecular hypergraph if it represents molecular structures, where $x \in V$ corresponds to an individual atom. Hyperedges with degrees greater than 2 corresponds to polycentric bonds and hyperedges with degree 2 corresponds to simple covalent bonds.

Hypergraph appear to be more convinent to describe some chemical structures . Hence the concept of molecular hypergraph may be seen as a generalisation of the concept of molecular graphs.

3.2 HYPERGRAPH THEORY FOR TELE COMMUNICATION

A hypergraph theory can be used to model cellular mobile communication systems. A cellular system is a set of cells where two cells can use the same channel if the distance between them is atleast some predefined value D.

The situation can be represented by a graph where:

• each vertex represents a cell.

• An edge exist between two vertices if and only if the distance between the corresponding vertices is less than the distance called the reuse distance and is denoted by D.

A forbidden set is a group of cells all which cannot use a channel simultaneously.

A minimal forbidden cell is a forbidden set which is minimal with respect to this property.ie,no proper subset of a minimal forbidden set is forbidden.

From these definitions it is possible to derive a better modelization using hypergraphs.
We proceed in the following ways:

- Each vertex represents a cell.

- A hyperedge is minimal forbidden set.

3.3 HYPERGRAPH THEORY FOR MODELLING PARALLEL DATA STRUCTURES

Hypergraph provide an effective means for modelling parallel data structures. A shared memory multi processor system consist of a number of processors and memory modules. We define a template as a set of data elements that need to be processed in parallel. Hence the data elements from a template should be stored in different memory modules.

So we can define a hypergraph in the following way:
- A data is represented by a vertex.
- Hyperedges are the templates.

From this model and by using the properties of hypergraphs one can resolve various problems such as the conflict - free - access to data in parallel memory system.

3.4 COMPLEX NETWORKS AS HYPERGRAPHS

The study of complex networks represents an important area of multidisciplinary research involving Physics,Mathematics,Chemistry,Biology,Social sciences and Information sciences,among others. These systems are commonly represented by means of simple or directed graphs that consist of sets of nodes representing the objects under investigation, e.g, People or groups of people, molecular entities ,computers etc; joined together in pairs by links if the corresponding nodes are related by some kind of relationships.These networks include the Internet, the World Wide Web, social networks, information networks, neutral networks food webs, reaction and metabolic network and Protein-Protein interaction networks.

In some cases the use of simple or directed graphs to represent complex network does not provide a complete description of the real world system under investigation. For instance in the collaboration network represented as a simple graph we only know whether the scientists have collaborated or not, but we cannot know whether three or more authors linked together in a network were coauthors of the same papers or not.

A possible solution to this problem is to represent the collaboration network as a bipartiate graph in which a disjoint set of nodes represent papers and another disjoint set represents author. However, in this case the homogenity in the definition of nodes is lost, because we have certain nodes that represent papers and others that represents authors.

A natural ay of representing these system is to use a generalization of graphs known as hypergraphs. In a graph a link relates only a pair of nodes, but here the edges of hypergraph ie; hyperedges can relate groups of more than two nodes. Thus we can represent the collaboration network as a hypergraph in which nodes represent authors and hyperedge represent group of authors that have published papers together. Despite the fact that complex weighted networks have been covered in some detail in physical literature, there are no reports on the use of hypergraphs to represent complex systems. Consequently we will formally introduce the hypergraph concept as generalization for representing complex networks and will call them complex hyper networks.

EXAMPLES OF COMPLEX HYPER NETWORKS

1. FOOD WEB

The trophic relations in ecological systems are normally represented through the use of food web, which are oriented graphs or digraphs whose nodes represent the species and links represent trophic relations between species. Another ay of representing food web is by means of competition graphs, which have the same set of nodes as the food web but in which two nodes are connected if and only if, the corresponding species compete for the same prey in the food web. In the competition graph we can only know if two linked species have common prey, but we cannot know the composition of the whole group of species tha compete for the common prey.

In order to solve this problem a competition hypergraph have been proposed in which nodes represent species in the food web and hyperedges represent groups of species that compete for common prey. It has been shown that many cases competition hypergraph yield more detailed description of the predation relations among the species in the food web than competition graphs.

CONCLUSION

We generalized several concepts related to connection in graphs to hypergraphs. While some of these concept generalize naturally in unique way, or behave in hypergraphs similarly to graphs, other concepts lead themselves to more than one natural generalization, or reveal surprising new properties. Many more concepts from graph theory remains unexplored for hypergraphs, and we hope that our work will stimulate more research in these area.

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