# THE FUNDAMENTAL GRAPH PRODUCTS

***A Dissertation submitted in partial fulfillment of the requirementfor the award of***

**DEGREE OF MASTER OF SCIENCE**

**IN MATHEMATICS**

***By***

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## CERTIFICATE

This is to certify that the dissertation entitled**" THE FUNDAMENTAL GRAPHPRODUCTS**" is a bonafide record of the work done by **Jyothy S**under my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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# DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of **Smt. Mary Runiya,** Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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# CONTENTS

Page No.

# INTRODUCTION

The primary object of interest in this project is the idea of a graph product. Every branch of mathematics employs some notion of a product that enables the combination or decomposition of its elemental structures. Broadly speaking, a graph product is a binary operation on the set of finite graphs. However , under reasonable and natural restrictions ( such as associativity), the number of different products is actually quite limited.

The product begins with definitions of three main products: the Cartesian product, the Direct product and the Strong product. In chapter 2, 3 and 4 we study about them in detail.

# PRELIMINARY

**Definition 0.1**: A simple graph is a set V(G) of vertices together with a set E(G) of unordered pair [u,v] of distinct vertices of G, the edges of G.

We represent graphically by drawing the vertices as nodes and the edges as line segments connecting the nodes. For an edge e=[u,v] we call u and v its endpoints and say e joins them.

**Definition 0.2**: Two or more edges of a graph with same end points are called parallel edges.

**Definition 0.3**: An edge with identical ends is called loop.

**Definition 0.4**: Two vertices u and v are adjacent when they are joined by an edge.

**Definition 0.5**: A vertex u adjacent to v is neighbor of v.

**Definition 0.6**: A graph is finite if its vertex set is finite and the set of finite simple graphs is denoted by Γ.

**Definition 0.7**: A graph G is called non-trivial if $|V(G)| > 1$.

**NOTE**

(1): $|V(G)|$ is known as the order and $|E(G)|$ as the size of the graph G.
(2): The set of finite graphs in which loops are admitted is denoted as $\Gamma_0$. Clearly $\Gamma \subset \Gamma_0$.

**Definition 0.8**: Given a vertex set v in a graph G, the neighborhood of v is defined as
$N(v) = \{u|uv\epsilon E(G)\}$
that is, the set consisting of all neighbors of v.

**NOTE**

If there is a loop at v, then $v\epsilon N(v)$.

**Definition 0.9**: A subgraph H of a graph G, we mean a graph H for which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

**Definition 0.10**: If all pairs of vertices of a subgraph H of G that are adjacent in G are also adjacent in H, then H is an induced subgraph.

**Definition 0.11**: A subgraph H of a graph G is a spanning subgraph if has the same vertex set as G, that is, if V(H)=V(G).

**Definition 0.12**: Two graphs G and H are called isomorphic, in symbol $G \cong H$, if there exist a bijection $\phi$ from V(G) to V(H) that preserves adjacence and non-adjacence, in other words, a mapping for which $[\phi(u), \phi(v)] \epsilon E(H)$ if and only if $[u, v] \epsilon E(G)$. Such a mapping $\phi$ is called an isomorphism between G and H.

**Definition 0.13**: $\phi$ is a homomorphism from a graph G into a graph H if it is an adjacency preserving mapping from V(G) into V(H), namely a mapping for which $|\phi(u), \phi(v)| \epsilon E(H)$ whenever $[u, v] \epsilon E(G)$.

**Definition 0.14**: A simple graph G is said to be complete if its each pair of distinct vertices is joined by an edge.
A complete graph with n vertices is usually denoted by $K_n$.

**Definition 0.15**: A graph G is called bipartite if its vertex set can be represented as the union of two disjoint sets $V_1$ and $V_2$ such that every edge of G connects an element of $V_1$ with one of $V_2$.
We call $V_1$ and $V_2$ bipartitions of V.

**Definition 0.16**: If every element of $V_1$ is adjacent to every element of $V_2$, then G is called a complete bipartite graph.
It is denoted by $K_{m,n}$, where m and n are cardinalities of $V_1$ and $V_2$.

**Definition 0.17**: A path $P_n$ is the graph whose vertices are 1,2,...,n and for which two vertices are adjacent precisely if their difference is $\pm 1$. A path in a graph G is a subgraph of G that is isomorphic to some $P_n$; in other words, a sequence of distinct vertices $V_1, V_2, ..., V_n$ in G where $V_i V_j$ is an edge of G whenever $i - j = \pm 1$.

**Definition 0.18**: A walk in G is a sequence of vertices $V_1, V_2, ..., V_n$ such that $V_i V_j \epsilon E(G)$ for i=1,2,...,n-1.

**Definition 0.19**: A graph without cycles is called acyclic.
Clearly, acyclic graphs are bipartite.

**Definition 0.20**: Connected acyclic graphs are called trees.

**Definition 0.21**: An isomorphism of a graph G onto itself is called automorphism. In other words, an automorphism of G is a permutation $\phi$ of $V(G)$ with the property that [u,v] is an edge if and only if $[\phi(u), \phi(v)]$ is an edge.

**Definition 0.22**: The complement $G^c$ of a simple graph G is the simple graph with vertex set V, two vertices being adjacent in $G^c$ if and only if they are not adjacent in G.

**Definition 0.23**: A simple graph G is self complement if G is isomorphic to $G^c$.

**Definition 0.24**: Let $G_1$ and $G_2$ be two subgraph of G, then $G_1$ and $G_2$ are disjoint if they have no vertex in common and edge disjoint if they have no edges in common.

**OPERATIONS ON GRAPH**

(1) Union of graphs: The union $G_1 \bigcup G_2$ of $G_1$ and $G_2$ is the subgraph with vertex set V($G_1 \bigcup G_2$) and edge set E($G_1 \bigcup G_2$).
(2) Intersection of graphs: The intersection $G_1 \bigcap G_2$ of $G_1$ and $G_2$ is the subgraph with vertex set v($G_1 \bigcap G_2$) and edge set E($G_1 \bigcap G_2$).

**Definition 0.25**: The degree of a vertex V in G is the number of edges of G incident with V, each loop counting as 2 edges.

**Definition 0.26**: A clique C, in an undirected graph G(V,E) is a subset of the vertices, $C \subseteq V$, such that every two distinct vertices are adjacent.

**Definition 0.27**: A weak homomorphism from G to H is a mapping $\phi$ from V(G) to V(H) with $\phi(x) = \phi(y)$ or $\phi(x)\phi(y) \epsilon E(H)$ for all $x, y \epsilon E(G)$.

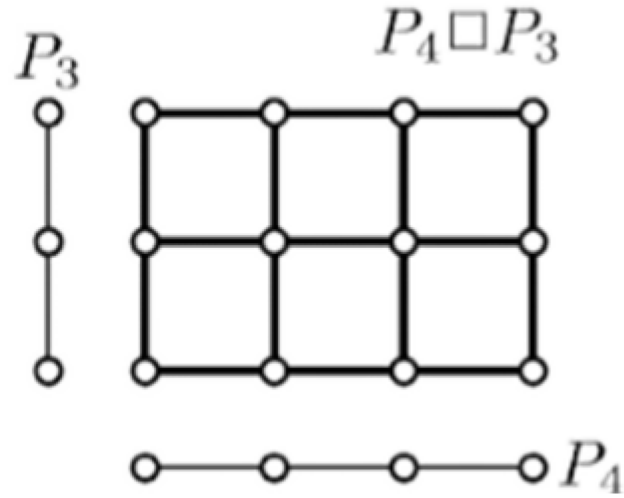
# Chapter 1

# FUNDAMENTAL GRAPH PRODUCTS

The three fundamental graph product :the Cartesian product, the Direct product and the Strong products. In each case, the product of graphs G and H is another graph whose vertex set is the Cartesian product $V(G) \times V(H)$ of sets. However, each product has different rules for adjacencies.

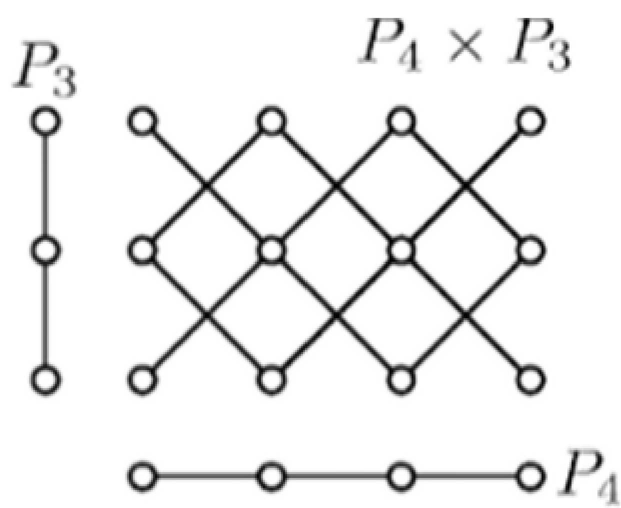
**Definition 1.1** *The Cartesian product of G and H is a graph, denoted as* $G \Box H$*, whose vertex set is* $V(G) \times V(H)$*. Two vertices (g,h) and (g',h') are adjacent precisely if g=g' and* $hh' \epsilon E(H)$*, or* $gg' \epsilon E(G)$ *and h=h'. Thus*
$V(G \Box H) = \{(g,h) | g \epsilon V(G) \text{ and } h \epsilon V(H)\}$
$E(G \Box H) = \{(g,h)(g',h') | g = g', hh' \epsilon E(H) \text{ or } gg' \epsilon E(G), h = h'\}.$

The graphs G and H are called factors of the product $G \Box H$.



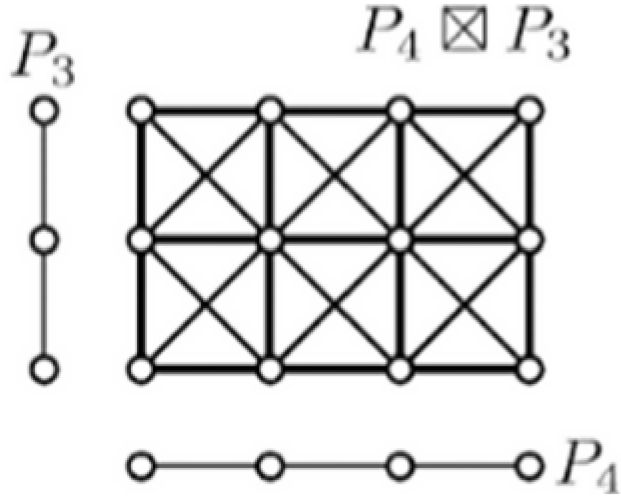

**Definition 1.2** *The Direct product of G and H is the graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$ and for which vertices (g,h) and (g',h') are adjacent precisely if $gg' \epsilon E(G)$ and $hh' \epsilon E(H)$. Thus*
$V(G \times H)=\{(g,h)|g\epsilon V(G)$ *and* $h\epsilon V(H)\}$,
$E(G \times H)$ $=\{(g,h)(g',h')|hh'\epsilon E(H)$ *or* $gg'\epsilon E(G)\}$.

Other names for the Direct product that have appeared in the literature are Tensor product, Kronecker product, Cardinal product, Relational product, Cross product, Conjuction, Weak Direct product.



**Definition 1.3** *The Strong product of G and H is the graph denoted by* $G \boxtimes H$ *and defined as*
$V(G \boxtimes H) = \{(g,h) | g \epsilon V(G) \ and \ h \epsilon V(H)\}$,
$E(G \boxtimes H) = E(G \Box H) \bigcup E(G \times H)$.

Occasionally one also encounters the names Strong Direct product or Symmetric composition for the strong product.




**NOTE**
1:$G \Box H$ and $G \times H$ are subgraphs of $G \boxtimes H$.
2:$K_m \boxtimes K_n = K_{mn}$.

**REMARK**

Observe that $K_2 \boxtimes K_2 = C_4$, $K_2 \times K_2 = K_2 + K_2$ and $K_2 \boxtimes K_2 = K_4$. This explains the rationale behind the symbols for the three products: The square of $K_2$ produces either the shape $\Box$, $\times$ or $\boxtimes$ depending on whether we use Cartesian, Direct or Strong product. The notation is due to Nesetril(1981).

## COMMUTATIVITY

It is immediate from the definitions of the three products that the map $(g, h) \rightarrow (h, g)$ is an isomorphism from $G * H$ to $H * G$, where $*$ stands for any one of the three fundamental products. Thus, the three products are commutative in the sense that $G * H \cong H * G$ for all graphs G and H.

## ASSOCIATIVITY

**Theorem 1.1** *The Cartesian, The Direct and The Strong product are each associative. In particular, given graphs $G_1$, $G_2$ and $G_3$, the map $((x_1, x_2), x_3) \rightarrow (x_1, (x_2, x_3))$ is an isomorphism $(G_1 * G_2) * G_3 \rightarrow G_1 * (G_2 * G_3)$, where $*$ stands for either The Cartesian, The Direct or The Strong product.*

**Proof**: For the Cartesian product, referring to its definition note that $((x_1, x_2), x_3)((y_1, y_2), y_3) \epsilon E((G_1 \Box G_2) \Box G_3)$ iff $x_i y_i \epsilon E(G_i)$ for exactly one index $i \epsilon \{1, 2, 3\}$ and $x_i = y_i$ for the other two indices. Similarly, the same condition characterize $(x_1, (x_2, x_3))(y_1, (y_2, y_3)) \epsilon E(G_1 \Box (G_2 \Box G_3))$. Thus, the map $((x_1, x_2), x_3) \rightarrow (x_1, (x_2, x_3))$ is indeed an isomorphism, so the Cartesian product is associative.

According to the definition of Direct product $((x_1, x_2), x_3)((y_1, y_2), y_3) \epsilon E((G_1 \times G_2) \times G_3)$ if and of if $x_i y_i \epsilon E(G_i)$ for each $i \epsilon \{1, 2, 3\}$. Similarly, the same condition characterize $(x_1, (x_2, x_3))(y_1, (y_2, y_3)) \epsilon E(G_1 \times (G_2 \times G_3))$. It follows that the map $((x_1, x_2), x_3) \rightarrow (x_1, (x_2, x_3))$ is an isomorphism from $(G_1 \times G_2) \times G)3$ to $G_1 \times (G_2 \times G_3)$ so the Direct product is associative.

Turning now to the Strong product, its definition gives $((x_1, x_2), x_3)((y_1, y_2), y_3) \epsilon E((G_1 \boxtimes G_2) \boxtimes G_3)$ if and only if $x_i y_i \epsilon E(G_i)$ or $x_i = y_i$ for each $i \epsilon \{1, 2, 3\}$ and $x_i \neq y_i$ for atleast one index i. Similarly, these same conditions characterize $((x_1, x_2), x_3)((y_1, y_2), y_3) \epsilon E(G_1 \boxtimes (G_2 \boxtimes G_3))$. Thus the Strong product is associative.■

**MULTIPLE FACTORS**

In general, given graphs $G_1, G_2, ..., G_k$, then $G_1 \Box G_2 \Box ... \Box G_k$ is the graph with vertex set $V(G_1) \times V(G_2) \times ... \times V(G_k)$, where two vertices $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ are adjacent if and only if $x_i y_i \epsilon E(G_i)$ for some index i and $x_i = y_j$ for $i \neq j$. We often use the notation $G_1 \Box G_2 \Box ... \Box G_k = \Box_{i=1}^{k} G_i$.

Generalising the Direct product to multiple factors $G_1 \times G_2 \times ... \times G_{_k} = \times_{i=1}^{k} G_i$ is the graph whose vertices is $V(G_1 \times V(G_2) \times ... \times V(G_k)$ and for which vertices $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ are adjacent if $x_i y_i \epsilon E(G_i)$ or $x_i = y_i$ for each index i.

Finally, $G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_{_k} = \boxtimes_{i=1}^{k} G_i$ has vertex set $V(G_1) \times V(G_2) \times ... \times V(G_k)$, and distinct vertices $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ are adjacent if and only if $x_i y_i \epsilon E(G_i)$ or $x_i = y_i$ for each $1 \leq i \leq k$.
We denote in general E$(G_1 \times G_2 \times ... \times G_{_k}) \neq E(G_1 \Box G_2 \Box ... \Box G_k) \bigcup E(G_1 \times G_2 \times ... \times G_k)$, unless k=2.

**NOTE**
The $k^{th}$ power of G with respect to the Cartesian product is denoted as $G^{\Box, k}$. Similarly we denote $k^{th}$ power of G with respect to the Direct product and Strong product as $G^{\times, k}$ and $G^{\boxtimes, k}$.

**PROJECTION**

Here we define the notation of projection from a product to its factors.

Let $*$ represent either the Cartesian, the direct or the Strong product operation. Consider a product $G_1 * G_2 * ... * G_k$. For any index $1 \leq i \leq k$, there is a projection map $p_i : G_i * G_2 * ... * G_k \rightarrow G_i$ defined as $p_i(x_1, x_2, ..., x_k) = x_i$. We call $x_i$ the $i^{th}$ coordinate of the vertex $(x_1, x_2, ..., x_k)$.

No matter which product $*$ represents, each projection $p_i$ is a weak homomorphism. Indeed, the definitions imply that if $(x_1, x_2, ..., x_k)(y_1, y_2, ..., y_k)$ is an edge of $G_i * G_2 * ... * G_k$ then either $x_i y_i \epsilon E(G_i)$ or $x_i = y_i$ for each $1 \leq i \leq k$, so each $p_i$ is a weak homomorphism. Even more is true for the Direct product, because $(x_1, x_2, ..., x_k)(y_1, y_2, ..., y_k)$ is an edge of $G_1 \times G_2 \times ... \times G_{_k}$ if and only if $x_i y_i \epsilon E(G_i)$ for each $1 \leq i \leq k$, each projection $p_i$ is actually a homomorphism.

# Chapter 2

# CARTESIAN PRODUCT

The Cartesian product of graphs, is a straightforward and natural construction. It has bee widely investigated, has numerous properties and is in many respects the simplest graph product.

Recall that if $G_1, G_2, ..., G_k$ are graphs, then their Cartesian product is the graph $G_1 \Box G_2 \Box ... \Box G_k = \Box_{i=1}^{k} G_i$ with vertex set $\{(x_1, x_2, ...x_k) | x_i \epsilon V(G_i)\}$ and for which two vertices $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ are adjacent whenever $x_i y_i \epsilon E(G_i)$ for exactly one index $1 \leq i \leq k$ and $x_i = y_j$ for each index $i \neq j$.

The Cartesian product is commutative and associative in the sense that the maps $(x_1, x_2) \to (x_2, x_1)$ and $((x_1, x_2), x_3) \to (x_1, (x_2, x_3))$ are isomorphisms.
$G_1 \Box G_2 \cong G_2 \Box G_1$
$(G_1 \Box G_2) \Box G_3 \cong G_1 \Box (G_2 \Box G_3)$.

It is immediate that the Cartesian product distributes over disjoint union.
$G_1 \Box (G_2 + G_3) = G_1 \Box G_2 + G_1 \Box G_3$.

Moreover, the trivial graph $K_1$ is a unit with respect to Cartesian product,ie,

$K_1 \Box G = G$ for any simple graph G.

**DISTANCE**

Distance in Cartesian product suggest that if G and H are paths, then $d_{G\Box H}((g,h),(g',h'))$=$d_G(g,g') + d_H(h,h')$. In fact, this is true for arbitrary G and H, according to the following theorem.

**Theorem 2.1** *If (g,h) and (g',h') are vertices of a Cartesian product $G\Box H$, then $d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$.*

**Proof**: First suppose that $d_G(g,g') = \infty$. Then G is a disjoint union $G = G_1 + G_2$ with $g \epsilon V(G_1)$ and $g' \epsilon V(G_2)$. Therefore, $G\Box H = (G_1 + G_2)\Box H = G_1\Box H + G_2\Box H$ with $(g,h)\epsilon V(G_1\Box H)$ and $(g',h')\epsilon V(G_2\Box H)$. Hence, $d_{G\Box H}((g,h),(g',h')) = \infty$. and the theorem follows. By identical reasoning, the theorem follows if $H(h,h') = \infty$.

Thus we assume that both $d_G(g,g')$ and $H(h,h')$ are finite. Let $P = a_0a_1a_2...a_{d_G(g,g')}$ be a path in G from $g = a_0$ to $g' = a_{d_G(g,g')}$. Let $Q = b_0b_1b_2...b_{d_H(h,h')}$ be a path in H from $h = b_0$ to $h' = b_{d_H(h,h')}$. This gives rise to two paths
$P \times \{h\} = (g,h)(a_1,h)(a_2,h)...(g',h)$
$\{g'\} \times Q = (g',h)(g',b_1)(g',b_2)...(g',h')$
in $G\Box H$ whose concatenation is a path of length $d_G(g,g')$+$d_H(h,h')$ from(g,h) to (g',h'). Hence $d_{G\Box H}((g,h),(g',h')) \leq d_G(g,g') + d_H(h,h')$.

Conversely, let R be a shortest path between (g,h) and (g',h'). Every edge of R is mapped into a single vertex by one of the projections $p_G$ or $p_H$ and into an edge by the other. This implies that $d_G(g,g') + d_H(h,h') \leq |E(p_G(R))| + |E(p_H(R))| = |E(R)| = d_{G\Box H}((g,h),(g',h'))$ and the proof is complete.■

In general, if $G = G_1\Box G_2\Box ...\Box G_k$ and x,y$\epsilon$ V(G), then
$d_G(x,y) = \Sigma_{i=1}^{k} d_{G_i}(p_i(x), p_i(y))$.

**PRIME FACTOR DECOMPOSITIONS**

The uniqueness of the prime factor decomposition of connected graphs with respect to the Cartesian product was first shown by Sabidussi (1960) and independently by Vizing (1963).

A graph is prime with respect to a given graph product if it is non-trivial and cannot be represented as the product of two non-trivial graphs. For the Cartesian product, this means that a non-trivial graph G is prime if G=$G_1 \square G_2$ implies that $G_1$ or $G_2$ is $K_1$.

**Theorem 2.2** *Every non-trivial graph G has a prime factor decomposition with respect to the Cartesian product. The number of prime factors is atmost* $log_2|V(G)|$

**Proof**: Because the product of k non-trivial graphs has atleast $2^k$ vertices, a graph G can have atmost $log_2|V(G)|$ factors. Thus there is a presentation of G as a product $G_1 \square G_2 \square ... \square G_l$ with a maximal number of factors. Clearly, every factor is prime.■

**BOX**

An important concept for all products are subproducts, which we call boxes. A box in a product $G = G_1 \square G_2 \square ... \square G_k$ is a subgraph of the form $G = U_1 \square U_2 \square ... \square U_k$, where $U_i \subseteq G_i$ for each index i.

**Lemma 2.1 (Unique Square Lemma)** Let e and f be two incident edges of a Cartesian product $G_1 \square G_2$ that are in different layers, that is, one in a $G_1$-layer and the other one in a $G_2$-layer. Then there exists exactly one square in $G_1 \square G_2$ containing e and f. This square has no diagonal.

**Proof**: We may assume e=uw=$(u_1, u_2)(v_1, u_2)$ and f=wv=$(v_1, u_2)(v_1, v_2)$. In particular, this means $u_1 \neq v_1$ and $u_2 \neq v_2$. Suppose z=$(z_1, z_2)$ is adjoint to both u and v. As z is adjacent to u=$(u_1, u_2)$, we have $z_1 = u_1$ or $z_2 = u_2$. As z is adjacent to v=$(v_1, v_2)$, we have $z_1 = v_1$ or $z_2 = v_2$. These constraints force either z=$(v_1, u_2) = w$ or $z = (u_1, v_2)$. We now have a unique (and diagonal free) square $(u_1, u_2)(v_1, u_2)(v_1, v_2)(u_1, v_2)$ containing e and f.■

We say a subgraph W of a Cartesian product G has the square property if for any two adjacent edges e,f that are in different layers, the unique square of G that contains e and f is also in W.

**Lemma 2.2** A connected subgraph W of a Cartesian product is a box if and only if it has square property.

**Proof**: By previous lemma all boxes have the square property. Suppose W is a connected subgraph of a Cartesian product G with square property. It suffices to prove the lemma for $G = G_1 \Box G_2$. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be vertices in W. We have to show that $(a_1, b_2)$ and $(b_1, a_2)$ are also in W. We may suppose that the vertices $(a_1, a_2), (b_1, a_2), (a_1, b_2)$ and $(b_1, b_2)$ are distinct. Because W is connected, there is a path P from a to b. Let us call an edge e of P a $G_1$-edge, if $p_2(e)$ consist only one vertex and a $G_2$-edge otherwise. By previous lemma, we can replace every sequence e,f of two edges in P, where e is a $G_1$-edge and f a $G_2$-edge, by two edges e',f' in W, where e' is a $G_2$-edge and f' a $G_1$-edge. Thus we can assume that P consist of a sequence of $G_1$-edges followed by a sequence of $G_2$-edges and that there also exists a path P' from a to b in W in which a sequence of $G_2$-edges is followed by a sequence of G-edges. But then the vertex $(a_1, b_2)$ is on P and $(b_1, a_2)$ on P', hence both in W.■

**Definition 2.1** *A subgraph $W \subseteq G$ is convex in G if every shortest G-path between vertices of W lies entirely in W.*

Convex subgraphs of Cartesian product have the square property.

**Lemma 2.3** A subgraph W of G=$G_1\Box G_2\Box ...\Box G_k$ is convex if and only if W=$U_1\Box U_2\Box ...\Box U_k$, where each $U_i$ is convex in $G_i$.

**Proof**: Suppose W is convex in G. Then it is connected and has the square property and it is a box by previous lemma. It follows that $W = p_1(W)\Box p_2(W)\Box ...p_k(W)$. We have to show that each $p_i(W)$ is convex. Fix i and take vertices $a_i$ and $b_i$ of $p_i(W)$. Let $x_i$be on a shortest $a_i, b_i$-path in $G_i$. We must show that $x_i$ belongs to $p_i(W)$.

Choose vertices $a = (a_1, a_2, ..., a_k)$ and $b = (b_1, b_2, ..., b_k)$ of W with $p_i(a) = a_i$ and $p_i(b) = b_i$. Define $x = (x_1, x_2, ..., x_k)$ as follows. For each $j \neq i$, let $x_j$ be an shortest $a_j, b_j$ path in $G_j$. Thus $d_{G_s}(a_s, b_s) = d_{G_s}(a_s, x_s) + d_{G_s}(x_s, b_s)$ for each $1 \leq s \leq k$. From this we get $d_G(a, b) = d_G(a, x) + d_G(x, b)$. It follows that x lies on a shortest a,b-path in G, so $x \epsilon W$ by convexity of W. Hence $x_i = p_i(W) \epsilon p_i(W)$.■

# Chapter 3

# DIRECT PRODUCT

We introduce the direct product in this chapter. Here we investigate its elementary properties in greater detail, deducing results analogous to those for the Cartesian in the previous chapter. Whereas the Cartesian and the Strong products are usually regarded as operations on the class of simple graphs $\Gamma$, the most natural setting for the Direct product is the class of graph $\Gamma_0$. Therefore, although we initially defined it as a product on $\Gamma$, we broaden our definition slightly, allowing it to graphs in $\Gamma_0$.

If $G_1, G_2, ..., G_K$ are graphs in $\Gamma_0$, then their Direct product is the graph $G_1 \times G_2 \times ... \times G_k = \times_{i=1k} G_i$ with vertex set $\{(x_1, x_2, ..., x_k) | x_i \epsilon V(G_i)\}$ and for which vertices $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ are adjacent precisely if $x_i y_i \epsilon E(G_i)$ for every $1 \leq i \leq k$.

As noted earlier the $k^{th}$ power of a graph G with respect to the Direct product is denoted as $G^{\times,k}$.

Recall that the Direct product is commutative and associative. Although our reasoning was then restricted to graphs in $\Gamma$, a review of the proof reveals that it remains valid in the class $\Gamma_0$. Thus the maps $(x_1, x_2) \to (x_2, x_1)$ and $((x_1, x_2), x_3) \to (x_1, (x_2, x_3))$ give rise to the following isomorphisms, where all factors belong to $\Gamma_0$.
$G_1 \times G_2 \cong G_2 \times G_1$

$(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$.

Clearly, the Direct product distributes over the disjoint union
$G_1 \times (G_1 + G_3) \cong G_1 \times G_2 + G_1 \times G_3$.

Let $G = G_1 \times G_2 \times ... \times G_k$. By simple wording of the definitions, each projection $p_i : G \to G_i$ is a homomorphism. Further, given a graph H and a collection of homomorphism $\phi_i : H \to G_i$ for $1 \leq i \leq k$, observe that the map $\phi : x \to (\phi_1(x), \phi_2(x), ..., \phi_k(x))$ is a homomorphism $H \to G$. From the two facts just mentioned, we see that every homomorphism $\phi : H \to G$ has the form $\phi : x \to (\phi_1(x), \phi_2(x), ..., \phi_k(x))$, for homomorphisms $\phi : H \to G_i$, where $\phi_i = p_i\phi$. Clearly $\phi$ is uniquely determined by the $p_i$ and $\phi_i$.

**DISTANCE**

The question of distance in Direct products, although simple, is somewhat more subtle than for other products. Consider the distance between vertices (g,h) and (g',h') in $G \times H$. Take a walk $W : (g, h)(a_1, b_1)(a_2, b_2)...(a_{n-1}, b_{n-1})(g', h')$ of length n joining these vertices. Because the projections are homomorphisms, it follows that $p_G(W) : ga_1a_2...a_{n-1}g'$ and $p_H(W) : hb_1b_2...b_{n-1}h'$ are g,g'- and h,h'-walks of length n in G and H respectively. Conversely, given walks $ga_1a_2...a_{n-1}g'$ in G and $hb_1b_2...b_{n-1}h'$ in H, both of length n, we can construct a walk $(g, h)(a_1, b_1)(a_2, b_2)...(a_{n-1}, b_{n-1})(g', h')$ of length n in $G \times H$.

**NON-UNIQUENESS OF PRIME FACTORIZATION**

The loop $K_1^s$ is a unit for the Direct product in the sense that $K_1^s \times G \cong G$ for every graph $G \epsilon \Gamma_0$.

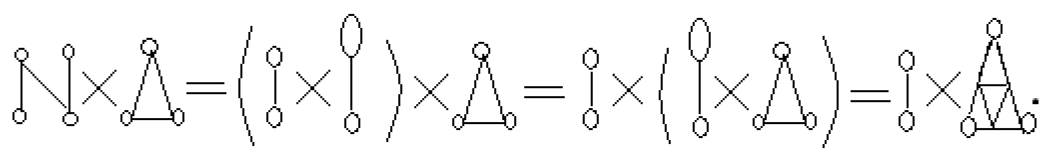
**Definition 3.1** *A graph G is prime with respect to the Direct product if it has more than one vertex and $G \cong G_1 \times G_2$ implies that either $G_1$ or $G_2$ equals $K_1^s$.*

**Definition 3.2** *An expression $G \cong G_1 \times G_2 \times ... \times G_k$, with each $G_i$ prime is called prime factorization of G.*

**Theorem 3.1** *Prime factorization with respect to the Direct product is not unique in*
*1: The class of graphs with loops at each vertex*
*2: The class of connected graphs in $\Gamma$*
*3: The class of connected graphs in $\Gamma_0$.*

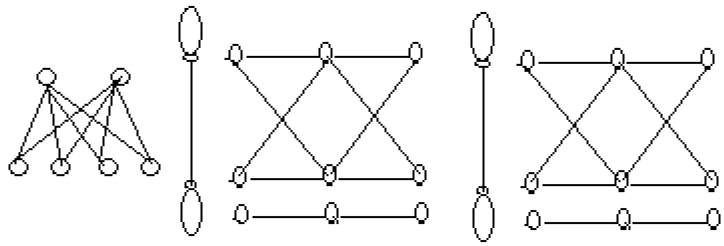
**Proof**: For the first assertation, let $K_p^s$ denote the graph obtained from $K_p$ by adding a loop at each vertex. Recall $G^{\times,n}$ denotes the $n^{th}$ direct power of G and the equation
$(K_1^s + K_2^s + (K_2^s)^{\times,2}) \times (K_1^s + (K_2^s)^{\times,3}) = (K_1^s + (K_2^s)^{\times,2} + (K_2^s)^{\times,4}) \times (K_1^s + K_2^s)$
Equality holds because both sides equal $(K_1^s + K_2^s + (K_2^s)^{\times,2} + (K_2^s)^{\times,3} + (K_2^s)^{\times,4} + (K_2^s)^{\times,5}$.
Clearly the factors involved are prime.

For (2), the following equation shows that connected graphs may have non-unique factorization in $\Gamma$.



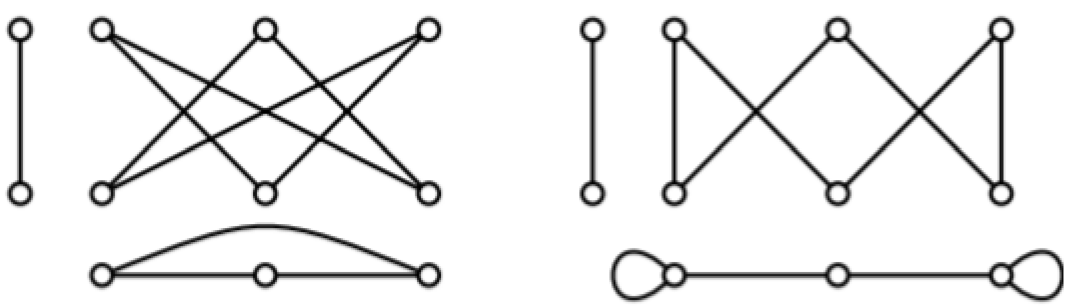
Here the graph $N \times \triangle$ on the left further factors into three terms in $\Gamma_0$. Applying the associate property and re-multiplying produces the graph $K_2 \times A$ on the right. Now, N is prime in $\Gamma$, as it could only factor non-trivially as a product of two vertices, yet $N = K_2 \times K_2$. Turning to other factors, $\triangle$ and $K_2$ are clearly prime. The graph A on the right is prime in $\Gamma$ because it has six vertices, so it could only factor as $A = K_2 \times G$ for some graph G with these vertices. But then A would be bipartite, a contradiction. Thus $N \times \triangle$ and $K_2 \times A$ are two different prime factorization (in $\Gamma$) of the same graph.

For (3), the figure below shows different prime factorizations of $C_6$ in $\Gamma_0$.



■

Having noted the failure of prime factorizations over the Direct product, we now consider the layers of the prime factorization of a connected graph over the Direct product. The following figure shows a graph that has different coordinatizations with respect to a prime factorization. In one case, the sets $\{1, a, 2\}$ and $\{3, b, 4\}$ are layers and in other case, sets $\{1, b, 2\}$ and $\{3, a, 4\}$ are layers.



A closer look at the figure reveals the source of the problem. Vertices a and b have the same neighborhood $\{1, 2, 3, 4\}$ so their positions in the factorizations can be interchanged.

Evidently, then, the existence of vertices with identical neighborhoods complicates the discussion of prime factorizations over the Direct product. To overcome this difficulty, we introduce a relation R on the vertices of a graph. Two vertices x and x' of a graph G in relation R, written as xRx, $N_G(x) = N_G(x')$.(For clarity, we occasionally write $R_G$ for R). Clearly the subgraph induced on a R-equivalence class is either totally disconnected or is a complete graph with loops at each vertex.

We denote the quotient (in $\Gamma_0$) of G by its R-equivalence classes as G/R. For example $G/R = K_2$ for the graph G in above figure. Also if G is a complete graph with loop at each vertex, then $G/R = K_1^s$.

**Definition 3.3** *A graph is called R-thin if all its R-equivalence classes contain just one vertex.*

Given $x \epsilon V(G)$, let $[x] = \{x' \epsilon V(G) | N_G(x') = N_G(x)\}$ denote the R-equivalence class containing x.

# Chapter 4

# STRONG PRODUCT

The Strong product was introduced in the first chapter. We now investigate its elementary properties in greater details.

Recall that if $G_1, G_2, ..., G_k$ are graphs in $\Gamma$, then their Strong product is the graph $G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_k = \boxtimes_{i=1}^{k} G_i$ with vertex set $\{(x_1, x_2, ..., x_k) | x_i \epsilon V(G_i)\}$ and for which two distinct vertices $(x_1, x_2, ..., x_k)$ and $(y_1, y_2, ..., y_k)$ are adjacent provided that $x_i y_i \epsilon E(G_i)$ or $x_i = y_i$ for each $1 \leq i \leq k$.

The Strong product is commutative and associative in the sense that the map $(x_1, x_2) \rightarrow (x_2, x_1)$ and $((x_1, x_2), x_3) \rightarrow (x_1.(x_2, x_3))$ are isomorphisms.
$G_1 \boxtimes G_2 \cong G_2 \boxtimes G_1$
$(G_1 \boxtimes G_2) \boxtimes G_3 \cong G_1 \boxtimes (G_2 \boxtimes G_3)$.

It is also immediate that the Strong product distributes over disjoint union.
$G_1 \boxtimes (G_2 + G_3) = G_1 \boxtimes G_2 + G_1 \boxtimes G_3$
Again, the trivial graph $K_1$ is a unit, that is, $K_1 \boxtimes G \cong G$.

Recall that each projection $p_i : G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_k \rightarrow G_i$ is a weak homomorphism. In other direction, given a graph H and a collection of weak homomorphisms $\phi_i : H \rightarrow G_i$, for $1 \leq i \leq k$, or observe that the map

$x \to (\phi_1(x), \phi_2(x), ..., \phi_k(x))$ is a weak homomorphism $H \to G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_k$. From the two facts just mentioned we see that every weak homomorphism $\phi : H \to G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_k$ necessary has form $x \to (\phi_1(x), ..., \phi_k(x))$ for weak homomorphism $\phi_i : H \to G_i$.

## DISTANCE

The distance in Strong products suggest that G and H are paths, then $d_{G\boxtimes H}((g,h),(g',h')) = max\{d_G(g,g'), d_H(h,h')\}$.

**Theorem 4.1** *If (g,h) and (g',h') are vertices of a Strong product* $G \boxtimes H$*, then* $d_{G\boxtimes H}((g,h),(g',h')) = max\{d_G(g,g'), d_H(h,h')\}$

**Proof**: First suppose $d_G(g,g') = \infty$. Then the graph G is a disjoint union $G = G_1+G_2$ with $g\epsilon V(G$ and $g'\epsilon V(G_2)$. Therefore, $G\boxtimes H = (G_1+G_2)\boxtimes H = G_1 \boxtimes H + G_2 \boxtimes H$, with $(g,h)\epsilon V(G_1 \boxtimes H)$ and $(g',h')\epsilon V(G_2 \boxtimes H)$. Hence $d_{G\boxtimes H}((g,h)(g',h')) = \infty$ and the theorem follows. By identical reasoning, the theorem holds if $d_H(h,h') = \infty$.

Thus assume that both $d_G(g,g')$ and $d_H(h,h')$ are finite. Let $P = a_0a_1...a_{d_G(g,g')}$ be a path in G from $g = a_0$ to $g' = a_{d_G(g,g')}$. Let $Q = b_0b_1...b_{d_H(h,h')}$ be a path in H from $h = b_0$ to $h' = b_{d_H(h,h')}$. By commutativity, we may assume $|P| \geq |Q|$. Consider the following two paths in $G \boxtimes H$:

$Q' = (g,h)(a_1,b_1)(a_2,b_2)...(a_{d_H(h,h')},h')$

$P' = (a_{d_H(h,h')},h')(a_{d_H(h,h')+1},h')...(g',h')$.

The concatenation of Q' and P' is a path of length $|P| = max\{d_G(g,g'), d_H(h,h')\}$ from (g,h) to (g',h'). Hence $d_{G\boxtimes H}((g,h)(g',h')) \leq max\{d_G(g,g'), d_H(h,h')\}$.

Conversely, because the projection $p_G$ is a weak homomorphism, it follows that

$d_G(g,g') = d_G(p_G(g,h), p_G(g',h')) \leq d_{G\boxtimes H}((g,h),(g',h'))$.

Also $d_H(h,h') \leq d_{G\boxtimes H}((g,h)(g',h'))$, so $max\{d_G(g,g'), d_H(h,h')\} \leq d_{G\boxtimes H}((g,h)(g',h'))$, and the proof is complete.$\blacksquare$

In general, if $G = G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_k$ and $x, y \epsilon V(G)$, then
$d_G(x,y) = max_{1\leq i\leq k}\{d_{G_i}(p_i(x), p_i(y))\}$.

## CLIQUES AND THE EXTRACTION OF COMPLETE FACTORS

The principal idea followed in this chapter for investigation the factorization of graphs over the Strong product is a careful examination of the mapping of cliques by isomorphisms between two composite graphs.

**Lemma 4.1** Let G and H be graphs and Q a clique of $G \boxtimes H$. Then $Q = p_G(Q) \boxtimes p_H(Q)$, where $p_G(Q)$ and $p_H(Q)$ are cliques of G and H respectively.

**Proof**: Clearly, $p_G(Q)$ and $p_H(Q)$ are complete subgraphs of G and H respectively. Thus there is a clique $Q_G$ of G with $p_G(x) \subseteq Q_G$ and a clique $Q_H$ of H with $p_H(Q) \subseteq Q_H$. Because $Q \subseteq Q_G \boxtimes Q_H$ and since Q is a maximal complete subgraph, we infer that $Q = Q_G \boxtimes Q_H$. Moreover $Q_G = p_G(Q)$ and $Q_H = p_H(Q)$.$\blacksquare$

**Theorem 4.2** *Two graphs G and H are isomorphic if and only if*
*(1): there is an isomorphism* $\Pi : G|S \to H|S$ *and*
*(2):* $|u| = |\Pi(u)|$, *for all* $u \epsilon V(G|S)$.

**Proof**: Suppose $\phi : G \to H$ is an isomorphism. Then $\phi([x]) = [\phi(x)]$, so $|[x]| = |[\phi(x)]|$. Define $\Pi : G|S \to H|S$ by $\Pi([x]) = [\phi(x)]$. But $|[\phi(x)]| = |[x]|$. This implies that $|\Pi[x]| = |[x]|$ for all $[x] \epsilon V(G|S)$. Thus $\Pi$ satisfies (1) and (2).

Conversely, suppose (1) and (2) hold. For $[x]\epsilon V(G|S)$, let $\phi_{[x]} : [x] \to \Pi([x])$ be a bijection. Then $\phi : V(G) \to V(H)$, defined by $\phi|[x] = \phi[x]$, is an isomorphism.■

**Lemma 4.2** For any integer $k \geq 1$, if $K_k \boxtimes G \cong K_k \boxtimes H$, then $G \cong H$.

**Proof**: Suppose there is an isomorphism $\phi : K_k \boxtimes G \to K_k \boxtimes H$. We just need to produce an isomorphism $\Pi : G|S \to H|S$ satisfying (2) of the above proposition. We know that S-classes of $K_k \boxtimes G$ have form $V(K_k) \times U$, for $U \epsilon V(G|S)$ and the S-classes of $K_k \boxtimes H$ have form $V(K_k) \times V$, $V \epsilon V(H|S)$. Thus $\phi$sends any S-class $V(K_k) \times U$ to an S-class $V(K_k) \times \Pi(U)$. Clearly $\Pi : G|S \to H|S$ is a bijection and $|U| = |\Pi(U)|$, as both cardinalities equal $|V(K_k) \times U|/K$. It is straightforward to check $\Pi$ is an isomorphism.■

## PRIME FACTORIZATION FOR CONNECTED GRAPHS

**Definition 4.1** *Let G be a graph and H' be a subgraph of H. The G-tower over H' is the subproduct $G \boxtimes H'$ of $G \boxtimes H$. More generally, if $G = G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_n$, we define $G_k$-tower in G as the product of $G_k$ by a subgraph of $G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_{k-1} \boxtimes G_{k+1} \boxtimes ... \boxtimes G_n$.*

**Lemma 4.3** Let $\phi : G \boxtimes H \to G' \boxtimes H'$ be an isomorphism and T be a G-tower over a clique Q of H. If G and H are connected, then $\phi(T)$ is a box of $G' \boxtimes H'$.

**Proof**: The Distance Formula implies that T is an isometric subgraph of $G \boxtimes H$, so $\phi(T)$ is an isometric subgraph of $G' \boxtimes H'$. This means that $\phi(T)$ is an induced subgraph.

It is straightforward to check that $\phi(T)$ is a subgraph of $p_{G'}(\phi(T) \boxtimes p_{H'}(\phi(T))$. We need to show $\phi(T) = p_{G'}(\phi(T)) \boxtimes p_{H'}(\phi(T))$. As $\phi(T)$ is induced, we just need to show that each vertex of $B = p_{G'}(\phi(T)) \boxtimes p_{H'}(\phi(T))$ belongs to $\phi(T)$.

Thus let (g,h) be a vertex of B. Then $\phi(T)$ must have vertices of the form (g,h') and (g',h). We will use induction on the distance from (g,h') to (g',h) to show that $(g,h)\epsilon\phi(T)$.

First suppose the distance is 1. Then the edge $\phi^{-1}(g,h')\phi{-}1(g',h)$ of T lies in some clique $Q'' \boxtimes Q$ in T, where Q" is a clique in G. Thus $\phi(Q'' \boxtimes Q) \subseteq \phi(T)$ is a clique in $G' \boxtimes H'$. By a lemma $\phi(Q'' \boxtimes Q)$ is a box in $G' \boxtimes H'$ and because this box contains the edge (g,h')(g',h), it also contains (g,h). Thus $(g,h)\epsilon\phi(T)$.

Now suppose $d((g,h'),(g',h)) > 1$. Because $\phi(T)$ is connected and isometric it has a shortest-path (g,h')(x,y)...(g',h). The induction hypothesis applied to paths (g,h')(x,y) and (x,y)...(g',h) yields $(g,y)(x,h)\epsilon\phi(T)$. This implies $d((g,y),(x,h)) < d((g,h'),(g',h))$ and the induction hypothesis once more gives $(g,h)\epsilon\phi(T)$.■

**Theorem 4.3** *Every connected graph has unique prime factor decomposition over the Strong product.*

**Proof**:Suppose a graph G has prime factor decomposition $G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_k$ and $G'_1 \boxtimes ... \boxtimes G'_l$. We may assume that $G_{r+1}, ..., G_k$ and $G'_{s+1}, ..., G'_l$ are complete and the other factors are not complete. Hence $G_1 \boxtimes G_2 \boxtimes ... \boxtimes G_r$ and $G'_1, ..., G'_s$ are not divisible be a non-trivial complete graph and thus
$G_1 \boxtimes ... \boxtimes G_r \cong G'_1 \boxtimes ... \boxtimes G'_s$
$G_{r+1} \boxtimes ... \boxtimes G_k \cong G'_{s+1} \boxtimes ... \boxtimes G'_l$.

As prime factorization is unique for complete graphs, the graphs $G'_{r+1}, ..., G'_k$ coincide with the graphs $G'_{s+1}, ..., G'_l$. Finally, by lemma (If $G \boxtimes H \cong G \boxtimes H'$ and G is prime but not complete, then $H \cong H'$) implies that the $G_1, ..., G_r$

coincide with the graphs $G'_1, ..., G'_s$.■.

# APPLICATION

The various real life applications of graph products are huge. A few of them are as follows:

(1) Graphs arising in Chemistry are a primary source of examples for graph theory: chemical trees, and fullerenes are just a few of the examples. After a molecule is represented as a graph, the primary goal of chemical graph theory is to investigate the graph and to predict the molecule's properties by computing carefully selected graph invariants. The Wiener index is the oldest such invariant.

(2) Another application includes a graph invariant called windex, introduced by Chung, Graham, and Saks in the context of dynamic location theory. It is closely connected to Cartesian products of complete graphs. These graphs are also known as Hamming graphs.

(3) Networks arise in many different areas, such as mathematical chemistry, software technology, and operations research. And, the investigation of very complex graphs and networks became an important research topic in the last decade, coinciding with increased interest in the Internet, citation networks and neural networks. To model large networks Kronecker graphs are used.

# CONCLUSION

In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects.

In this project we have studied about the fundamental graph products and some of its properties including commutativity, associativity, distance between vertices, prime factorization. The purpose of the project is to provide a brief knowledge about the three main graph products. We have also considered some of the applications of graph products.

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