INTRODUCTION TO GRACEFUL GRAPHS

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CERTIFICATE

This is to certify that the dissertationentitled "INTRODUCTION TO GRACEFUL GRAPHS" is a bonafide record of the work done by ARCHANA C.Kunder my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St.Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

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I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Ms. MARY RUNIYA, Assistant Professor, Department of Mathematics, St Teresa's College (Autonomous) Ernakulam and has not been included in any other project submitted previously for the award of any degree.

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I bow my head before God Almighty who showered His abundant grace on me to make this project a success.

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I am most graceful to the Library staff for the help extended in accomplishing this work.

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ERNAKULAM | **ARCHANA C.K**

APRIL 2018 | **REGISTER NO: SM16MAT005**

INTRODUCTION

The study of graph labelings has become a major subfield of graph theory. Very often the problems from this area drew attention due to their applications to real life situations or in some cases their history. Many of the most ardour problems of graph theory are easiest to state

Here we investigate another some what longstanding problem in graph labeling . The Graceful tree conjecture which states that every tree graph on n vertices has some vertex labeling using the numbers 1,2,3,...,n such that the edge labeling obtained from the vertex of the difference of 2 adjacent vertex labels assigns distinct edge labels.

The famous conjecture says that every tree is graceful but it has not yet been possible to prove it for any trees or come up with a counter example .

Alexander Rosa was the first to consider such a labeling scheme as a method to prove the conjecture and established that these conjectures would be solved by showing that every tree graceful. The problem is one of the most researched problem as there have been more than 500 paper on this in the last 50 years.

The conjecture has been tried with mostly 2 ways which are newer classes of trees are proved to be graceful or device a common programming model for generating graceful labels.

Here in this paper there are some classes of trees that has been gracefully labrled and some applications of gracefully labeled trees.

CHAPTER 1

PRELIMINARIES

A graph G is an ordered pair (V ,E) where V is a set of elements called vertices and E is a set of unordered pairs of distinct vertices from V called edges.
We say an edge e connects 2 vertices u and v denoting as e= uv.
And we say u and v are adjacent if they are connected by an edge.
The set of adjacent vertices of a vertex u is denoted as N(u) and it is also called the set of neighbours of u. The degree of a vertex u is d(u) = $|N(u)|$, the number of neighbours of u.

For a given graph G , when the vertex set and the edge set are not given explicitly we refer to them as V(G) and E(G) and we use the letters n and m as the number of vertices and edges respectively.
A sub graph H of G is a graph such that V(H) $\subset$ V(G) and E(H) $\subset$ E(G)

For a graph G =(V,E) and a subset W$\subseteq$ V ,the subgraph of G induced by W , denoted as G(W) is the graph H =(W,F) such that for all u,v $\in$ W if uv $\in$ E, then uv $\in$ F.
We say H is an induced subgraph of G. H is an induced subgraph of G if it is obtained by deletion of vertices and H is a subgraph of G if it is obtained by deletion of vertices and edges.

A walk in a graph is a finite sequence of vertices W = $(v_0, v_1, v_2...v_k$) such that $v_i v_{i+1}$ is an edge of the graph.

If the waik W does not go through an edge twice , we say W is a trail and if does not go through a vertex twice, we say W is a path.
A path starting in u and ending in v is called a uv path.

The length of a path is the number of edges and distance between 2 vertices u and v is the length of the shortest path between them and deenoted as dist (u,v) .
If there is no path between u and v then dist(u,v) $= \infty$

A graph is said to be connected if every pair of vertices is connected by a path.
If there is exactly one path connecting each pair of vertices , we say G is a tree.
Equivalently , a tree is a connected graph with n-1 edges.

A path graph P_n is a connected graph on n vertices such that each vertex has degree at most 2.
A cycle graph C_n is a connected graph on n vertices such that every vertex has degree 2.

A complete graph K_n is a graph with n vertices such that every vertex is adjacent to all others. On the other hand ,an independent set is a set of vertices of a graph in which no vertices are adjacent.
We denote I_n for an independent set with n vertices.

A bipartite graph G =(V ,E) is a graph such that there exists a partition P = (A,B) of V such that every edge of G connects a vertex in A to one in B.
Equivalently ,G is said to be bipartite if A and B are independent sets.
The bipartite graph is also denoted as G =(A ,B, E)

The join of 2 graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and with disjoint vertex sets is the graph G = (V,E) such that V = $V_1 \cup V_2$ and E = $E_1 \cup E_2 \cup \{uv|u \in V_1, v \in V_2\}$

Finally for a given graph G = (V ,E) a vertex labeling of G is a function f: V $\longrightarrow$ N and an edge labeling of G is a function g : E $\longrightarrow$ N.
Intutively we are assigning labels to vertices and \ or edges of the graph.
Throughout we have the co domains as a finite subset of N and we denote [a, b] = $\{a, a+1, ...b\}$.

CHAPTER 2

A graph is a collection of nodes and lines that we call vertices and edges respectively. A graph can be labeled or unlabeled.In many labeled graphs, the labels are used for identification only.Labeling can be used not only to identify vertices and edges ,but also to signify some additional properties depending on the particular type of labeling.

A labeling or valuation of a graph G is an assignment f of labels from a set of non negative integers to the vertices of G that induce a label for each edge xy defined by the labels f(x) and f(y).

GRACEFUL LABELING

A graceful labeling of a simple graph G(V,E) is a vertex labeling f: V(G) $\longrightarrow$ {0,1,2,3,...$|E|$ } such that f is injective and the edge labeling f_γ : E(G) $\longrightarrow${1,2,3,...$|E|$ } defined by f_γ(uv) =$|f(u) - f(v)|$,where uv ϵ E(G) is also injective. If a graph G admits a graceful labeling we say G is a graceful graph.

A labeled graph which can be gracefully numbered is called a graceful graph.

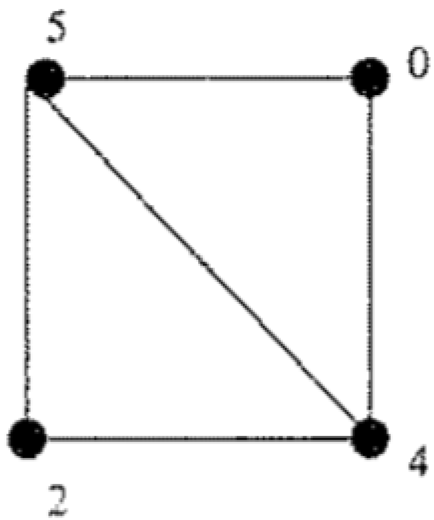
In order to achieve this

- Label the nodes (vertices) with distict non negative integers say between 0 and e both inclusive where e is the no of edges

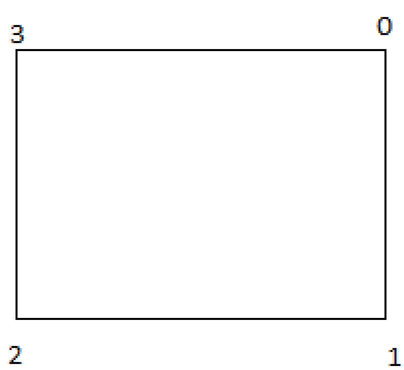
- Label the graph edges with the absolute differences between node values. If the graph edge numbers then run from 1 to e, the graph is gracefully numbered

- In order for a graph to be graceful , it must be without loops or multiple edges.

Example of a gracefully labelled graph



Example of a labelled graph not graceful



Although numerous families of graceful graphs are known ,a general necessary or sufficient condition for gracefulness has not yet been found. Also it is not known if all the tree graphs are graceful.

Most graph labeling methods trace their orgin from the definition introduced by Rosa. The concept of a β valuation was introduced by Rosa in 1966. Rosa called a function f a β -valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, 3, ...q\}$ such that ,when each edge xy is assigned the label $|f(x) - f(y)|$,the resulting edge labels are distinct . Golomb subsequently called such labeling as graceful . And this name was popularised by mathematician Martin Gardner. This terminology is now most commonly used.

Graceful labeling orginated as means of attacking the conjecture of Ringel that K_{2n+1} can be decomposed into 2n+1 subgraphs that are all isomorphic to a given tree with n edges.

Most graphs that have some sort of regularity of structure are graceful.Sheppard has shown that there are exactly q! gracefully labeled graphs with q edges.

Rosa has idenified 3 reasons when a graph fails to be graceful.

- G has "too many vertices and "not many edges".
- G has " too many edges".
- G satisfies Parity Condition

Rosa developed a useful parity condition for a simple graph G with e edges. He proved that if every vertex of G has even degree and if e $\equiv$ 1 or 2 (mod 4) ,then G is not graceful. Rosa's parity condition serves as a sufficient condition for non graceful graphs.

In 1982, Acharya proved that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph. In 2008, Acharya ,Rao and Arumugam proved that every tree can be embedded as an induced subgraph of a graceful tree.

Golomb gives sn application of graph labeling and gives a condition for the labeling to be graceful. Suppose we are given solid bar of integer length l. We want to carve n-2 notches in it at integer distances from either end with the condition that the distance between any 2 notches as well as the distances from any notch to either of the endpoints ,are all distinct.

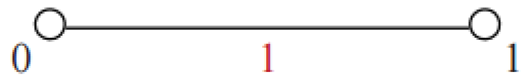
This problem can be viewed as labeling the vertices of athe complete graph K_n with the elements of the set $\{0, 1, 2, 3, ...l\}$ and labeling each edge by taking the absolute value of the difference of its end vertices. The problem is to find the smallest integer l so that K_n can be labeled this way. In the event that the vertices can be labeled using only numbers from N = $\{0, 1, 2, 3, ... \binom{n}{2}\}$ with each edge label being distinct , then each label being distinct , then each number from $N|\{0\}$ is used once. But this simply implies that for this n ,K_n has a graceful labeling .

EXAMPLE

Looking at the complete graphs on n vertices for n =2, 3 ,4 , we have the vertex labelings shown in figure are graceful .

It turns out that n = 4 is the largest n for which K_n can be labeled gracefully as proven by Golomb.

Gracefully labeled K_n for n =2,3,4


0
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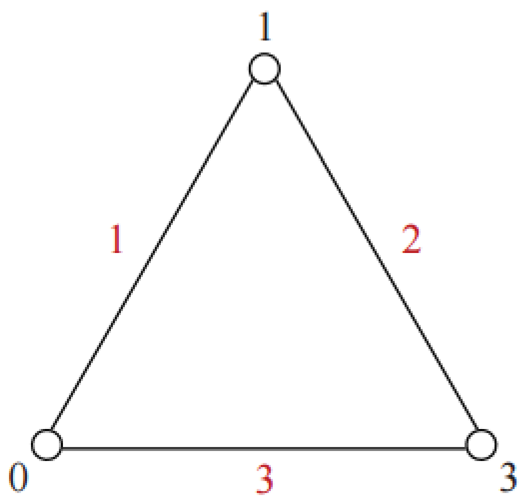
1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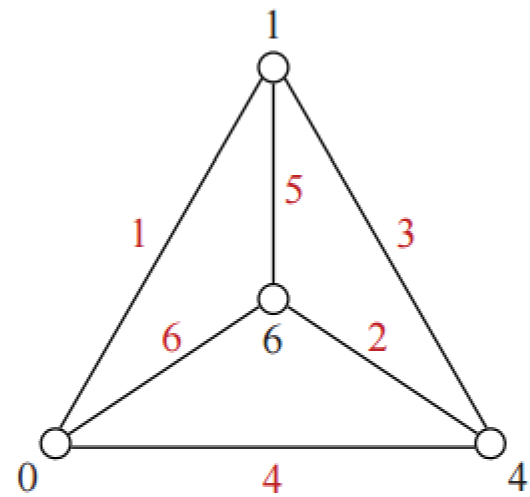
1
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6
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THEOREM 1

If n> 4 , the complete graph K_n is not graceful.

Proof

We begin by noting that for n> 4 , m $=|E(K_n)| = \binom{n}{2} \geq 10$

Suppose to the contrary that K_n is graceful.

Then V =V(K_n) $\subseteq \{0, 1, 2, 3, ...m\}$ and the edges have distinct labels with E = E(K_n) $=\{1, 2, 3, ...m\}$. For K_n to have an edge labeled m, we have that 0 and m must be vertices. Since $|m - o|$ = m is an edge label . In order to have an edge labeled m-1 , either 1 or m-1 is also a vertex label. For any graceful graph G with m edges, the replacement of every vertex label a_i with m-a_i does not change the edge labels .Thus we can choose the vertex label 1 for K_n instead of m-1 with no loss of generality.

Now to get an edge labeled m-2 , we have to adjoin the vertex label m-2. This is because in order to get the edge label m-2 from the difference of m-1 and 1 ,we would have 2 edges labeled 1, one between vertices 0 and 1 ,and the other between m-1 and m.If a vertex labeled 2 is added to get an edge labeled m-2 as the difference of m and 2, again we get 2 edges labeled 1 ,one between vertices 0 and 1, and the other between vertices 1 and 2.

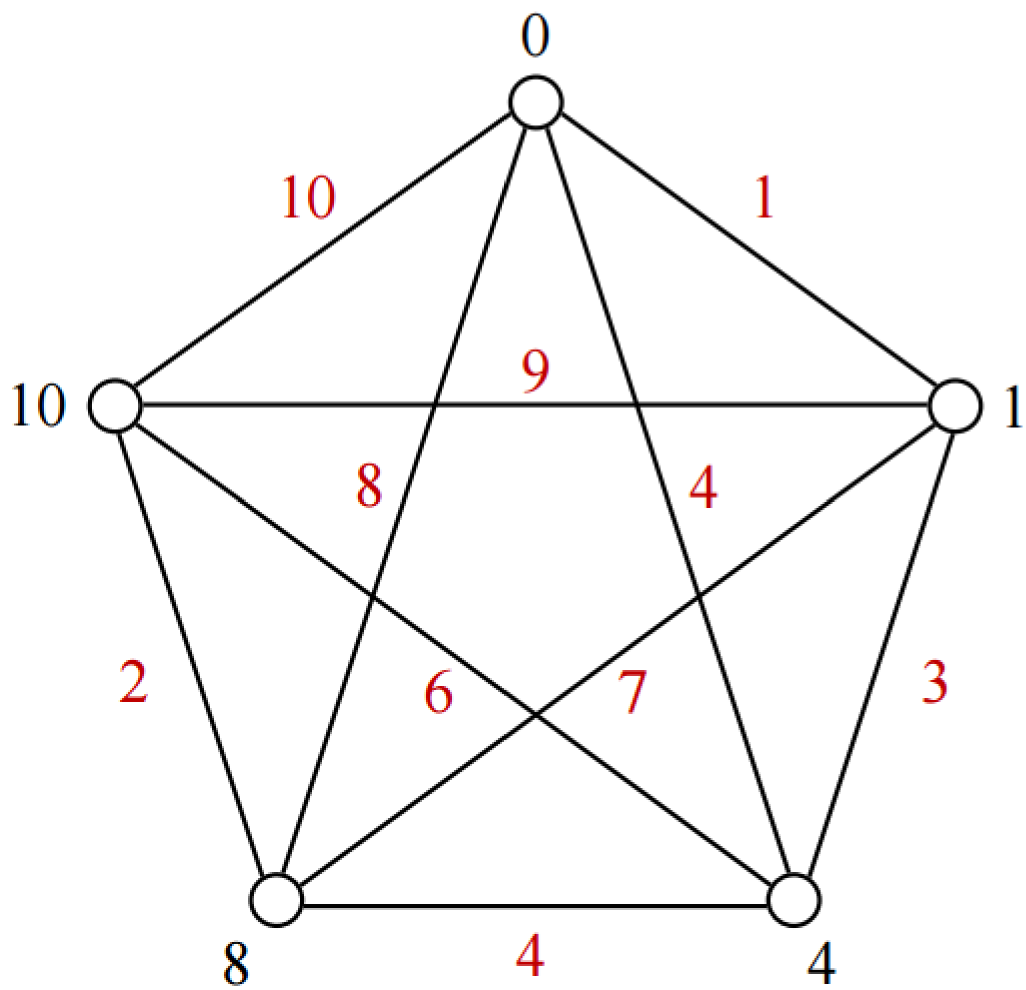
With vertices having labels 0,1,m-2 and m we get edges labeled 1,2,m-3 ,m-2 ,m-1 and m. In order to have an edge labeled m-4 , we must use 4 as a vertex label.This is because if we use 2, then we get 2 edges labeled 2 as a difference of 2 and 0 and as a difference of m-2 and m.

If n =5 then we should be done (we do have precisely 5 vertex labels). However , in this case m =10 and we have m-6 =4 ,so there actually is a pair of edges that have the same induced label as shown in the below figure.Thus for n=5, there is no good labeling . we need to only consider the case when n> 5, In this case m> 10 and so 4< m-6 and there clearly is no repetition of edge labels at this point . Moreover ,we still need more labels. With vertices having labels 0,1,4,m-2 and m , we have edge labels 1,2,3,4,m-6,m-4, m-3, m-2 ,m-1 and m. Note that for K_4, m=6.

There is no way to obtain an edge labeled m-5 because all choices for ways to get such an edge label contains at least one vertex label that can't be used. Adding a vertex labeled m-5 gives a second edge labeled m-6 = (m-5) -1. Adding a vertex labeled m-4 gives 2 edges labeled 4 = m- (m-4). Adding a vertex labeled m-1 gives duplicate edges labeled m-2 =(m-1) -1 . We cannot add a vertex labeled 3 since we get 2 edges labeled 3-1 =2.

Finally , our last choice is adding a vertex labeled 5. This is not possible either sincxe we get 2 edges labeled 5-4 =1 .This is a contradiction to our assumption that K_n is graceful for all cases when m-5 > 4, which corresponds to n≥ 5.

K_5 with repeated labels


0
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We start by showing a couple of results concerning necessary condition to the existence of a graceful labeling of a graph.
Rosa also gives a necessary condition for a graph to be graceful.

THEOREM 2

If G is an Eulerian graph with m edges such that m$\equiv$ 1 or 2 (mod 4) , then G cannot be labeled gracefully.

Proof

Suppose to the contrary that G is graceful graph of size m$\equiv$ 1 or 2 (mod 4) with an Eulerian circuit , and suppose G is graceful. Taking the sum over all edge labels b_i , we have

$$\sum_{i=1}^{m} bi = \sum_{i=1}^{m} i = \frac{(m+1)m}{2}$$

Now it is easy to see that if m$\equiv$ 1 or 2 (mod 4) then $\sum_{i=1}^{m} b_i \equiv 1 \pmod 2$.

However , for an arbitrary closed path C = a_1 ,a_2 ,a_3 ... a_m,a_1 , the edge labels are given by $|a_1 - a_2|$,$|a_2 - a_3|$, ... ,$|a_m - a_1|$ and as $|y| \equiv$ y (mod 2), the sum of the edge labels , with the indicies taken modulo m+1 is

$$\sum_{i=1}^{m} |a_i - a_{i+1}| = \sum_{i=1}^{m}(a_i - a_{i+1}) \equiv 0 \pmod 2$$

Thus we have reached a contradiction and the claim is proven.

The next one is a straight forward condition given by Golomb.

PROPOSITION 1

If G = (V,E) is graceful , then there exists a partition P =(A,B) of V such that the number of edges with one end in A and the other in B is $[\frac{m}{2}]$.

PROOF

Let G = (V,E) be a graph with a graceful labeling f and consider the partition P =(A,B) of V such that A = $\{u \in V; f(u) \equiv 0(mod2)\}$. Since there are $[\frac{m}{2}]$ odd vertices between 1 and m , and an odd difference is onll possible by substracting an even value from an odd one, the number of edges connecting 2 vertices with different parities must be exactly $[\frac{m}{2}]$

THE GRACEFUL TREE CONJECTURE

In the graph theory ,a major unproven conjecture is the Graceful Tree Conjecture or Ringel - Kotzig Conjecture , named after Gerhard Ringel and Anton Kotzig.

The conjectures of Ringel and Kotzig have led to one of the most easily stated , yet elusive conjectures in the realm of graph labelings.

The conjecture hypothesizes that "All trees are graceful." The Ringel - Kotzig conjecture is also known as the 'Graceful Labeling Conjecture. To prove the conjecture Kotzig once called the effort a disease. Many methods have been developed in the hopes of residing the nearly 50 year old problem .So far no proof of the truth or falsity of the conjecture has been found. Initially Rosa established the gracefulness of several classes of trees. Since then other classes have been shown to admit graceful labelings. We will investigate these classes.

THEOREM 2

If a graph G can be gracefully labeled by labeling the vertices from the set of integers , then G can be gracefully labeled by labeling the vertices from the set of non negative integers.

PROOF

Let G be gracefully labeled graph , with the vertices labeled from the set of all integers .
Call the smallest integer k.
Subsrtract k from every vertex labeling.
The smallest vertex labeling is now k-k=0 .So all vertices are labeled with non negative integers.
For any 2 vertices u,v $\in$ V (G) the edge uv orginally had the value $|f(u) - f(v)|$
The edge uv now has value $|f(u) - k - (f(v) - k)|$ =$|f(u) - k - f(v) + k|$ =$|f(u) - f(v)|$. Thus the edge values are preserved .So this is still a graceful labeling.

THEOREM 3

IF a graph G can be gracefully labeled by labeling the vertices from the set of integers , then G can be gracefully labeled by labeling the vertices from the set of positive integers .

CONJECTURE 1

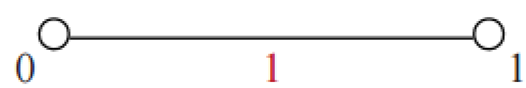
If a (p,q) graph G can be gracefully labeled by labeling the vertices from the set of integers , then G can be gracefully labeled by labeling the vertices from the set $\{0, 1, 2, 3, ...q\}$.
Unfortunately , this is still a conjecture.
The importance of this is that if this conjecture is true , we will be able to prove that all trees are graceful.

THEOREM 4

If the fact that a (p,q) graph G can be gracefully labeled by labeling the vertices from the set of integers implies that G can be gracefully labeled by labeling the vertices from the set $\{0, 1, 2, 3, ...q\}$, then all non trivial trees are graceful.

PROOF

We use induction on q .
Base case: When q = 1 , then we are done.



Induction Hypothesis :
Assume every non trivial tree with q edges is graceful. Now look at tree G with q+1 edges.

G is a tree , so has a vertex of degree 1 , call it v.
Now look at G-v
V only has degree 1, so deleting v is only removing one edge from G, call it edge e
So G-v has q edges.
A veertex of degree 1 cannot be a cut vertex, so since G is connected (since it is a tree) , G-v is connected.
G has no cycles (since it is a tree) , so G-v has no cycles.
So G-v is a tree with q edges.
So by our induction hypothesis, G-v is graceful.
So the vertices of G-v can be labeled gracefully from the set $\{0, 1, 2, 3, ...q\}$, with the edges of G-v having values 1,2,3,...q.
Now look again at G.
Keep all the vertices (except v) labeled as they were in the graceful labeling of G-v.
Thus the edges of G (except edge e) have values 1,2,3,...q.
We know edge e is incident to v, so let uv be edge e.
u is already labeled some integer from the set $\{0, 1, 2, 3, ...q\}$, call the integer u is labeled k.Label vertex v with k+q+1.
This is legal since all the other vertices of G are labeled from the set $\{0, 1, 2, 3, ...q\}$ and k+q+1 $>$ q, so no other vertex has this label.
Then edge e has value $|(k + q + 1) - k| = |q + 1|$ = q+1
Therefore the edges of G have values 1,2,3,...q,q+1.
So the vertices of G are labeled with distinct integers, and the edges have values 1,2,3,...q+1.
Thus G is graceful.

CHAPTER 3

SEVERAL CLASSES OF GRACEFUL TREES

PATHS

A graph is called a path (fig) if the degree d(v) of every vertex v is ≤ 2 and there are not more than 2 end vertices. An end vertex or leaf is a vertex of degree 1.

THEOREM 5

Every path is graceful.

PROOF

We demonstrate an algorithm to gracefully label any path P_n with n vertices.

In a path the no of edges is one less than the no of vertices or m = n-1.
Labeling can begin at either end without loss of generality.
The first vertex at one end is labeled 0, the adjacent vertex is labeled n-1 ,the next adjacent , non labeled vertex is labeled 1 and we continue in this manner.

Alternate vertices are incremently increasing by 1 while the remaining vertices are incremently decreasing by 1.
Consider the 2 cases where n $\equiv$ 0 (mod 2) and n $\equiv$ 1(mod 2) as shown in figure A below.
In both cases k =$[\frac{n}{2}]$
For even case , the edge labels begining with the leftmost edge in figure A are $|(n-1)-0|$, $|(n-1)-1|$, $|(n-2)-1|$,..., $|(n-k)-(k-1)|$ = n-1,n-2,n-3...1
In determining the last edge value recall k =$[\frac{n}{2}]$
For cases when n is even k =$[\frac{n}{2}]$ =$\frac{n}{2}$ and (n-k) -(k-1) = n-k-k+1 =1

It is easy to see this is a graceful labeling since all numbers between 1 and n-1 or m are used in the edge labels.
Similarly when n is odd the edge values beginning on the left are n-1, n-2, n-3 ,...1.
In evaluating the right most edge value recall k =$[\frac{n}{2}]$ =$\frac{n-1}{2}$ where n is odd.
Then n-k-k =n - $\frac{n-1}{2}$ - $\frac{n-1}{2}$ $= 1$
Again every value from 1 to n-1 or m is used .

GRACEFULLY LABELLED PATH P_n WHERE n EVEN

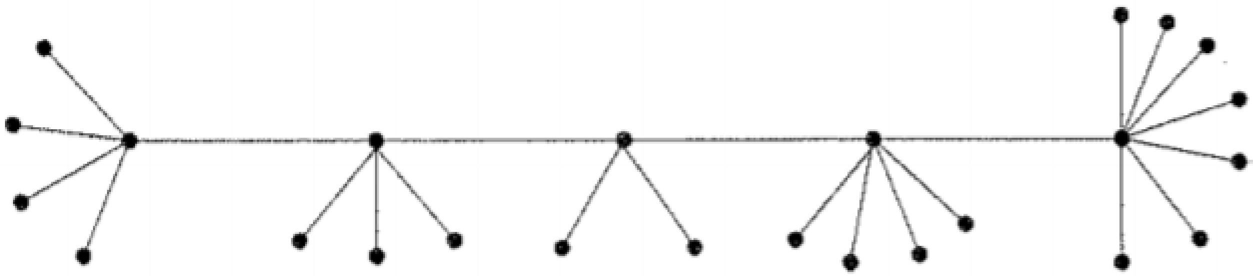
GRACEFULLY LABELLED PATH P_n WHERE n ODD

That is Every path is graceful.

CATERPILLAR

A caterpillar or a caterpillar tree in which all the vertices are within distance 1 of a central path.
Caterpillar trees have been used in chemical graph theory to represent the structure of benzenoid hyrocarbon molecules. In this representation one forms a caterpillar in which each edge corresponds to a 6 Carbon ring in the molecular strucure, and two edges are incident at a vertex whenever the corresponding rings belong to a sequence of rings connected end to end in the structure.

Figure shown below.



Now it follows that a caterpillar has atleast 2 vertices and if it has atleast 3 vertices, then the central path exists and is unique.
The next theorem establishes the gracefulness of caterpillars.

THEOREM 6

All caterpillars are graceful.

PROOF

Let T be a catterpillar with n vertices.
If n $\leq$ 2, T then T is a path of length one and we are done.So we assume n$\geq$ 3, and let P= $v_0, v_1, v_2...v_k$ be a path obtained from T by deleting the leaves of n $\geq$ 3 , P has atleast one vertex .

We now partition the vertex set of T as follows:
X $=\{x|x \in\ V(T), d(v_0, x) \equiv 0(mod2)\}$

Y $=\{y|y \in\ V(T), d(v_0, y) \equiv 1(mod2)\}$

=V(T) -X.

We note that $v_i \in$ X if i is even and $v_i \in$ Y if i is odd.
Label v_0 with n. Label the neighbours of v_0 with 1,2,3,... where the neighbour getting the largest label is v_1 .Assign labels n-1, n-2 ,n-3, ... to the neighbours of v_1 , other than v_2 .Continue as follows.

After v_{2i} receives its label, assign increasing integer labels to its neighbours other than v_{2i-1} starting with the smallest unused label , assigning the largest label to v_{2i+1} . Then assign labels to the neighbours of v_{2i+1} , other than v_{2i} , in decreasing order starting with the largest unused integer smaller than n ending by labeling v_{2i+2}.

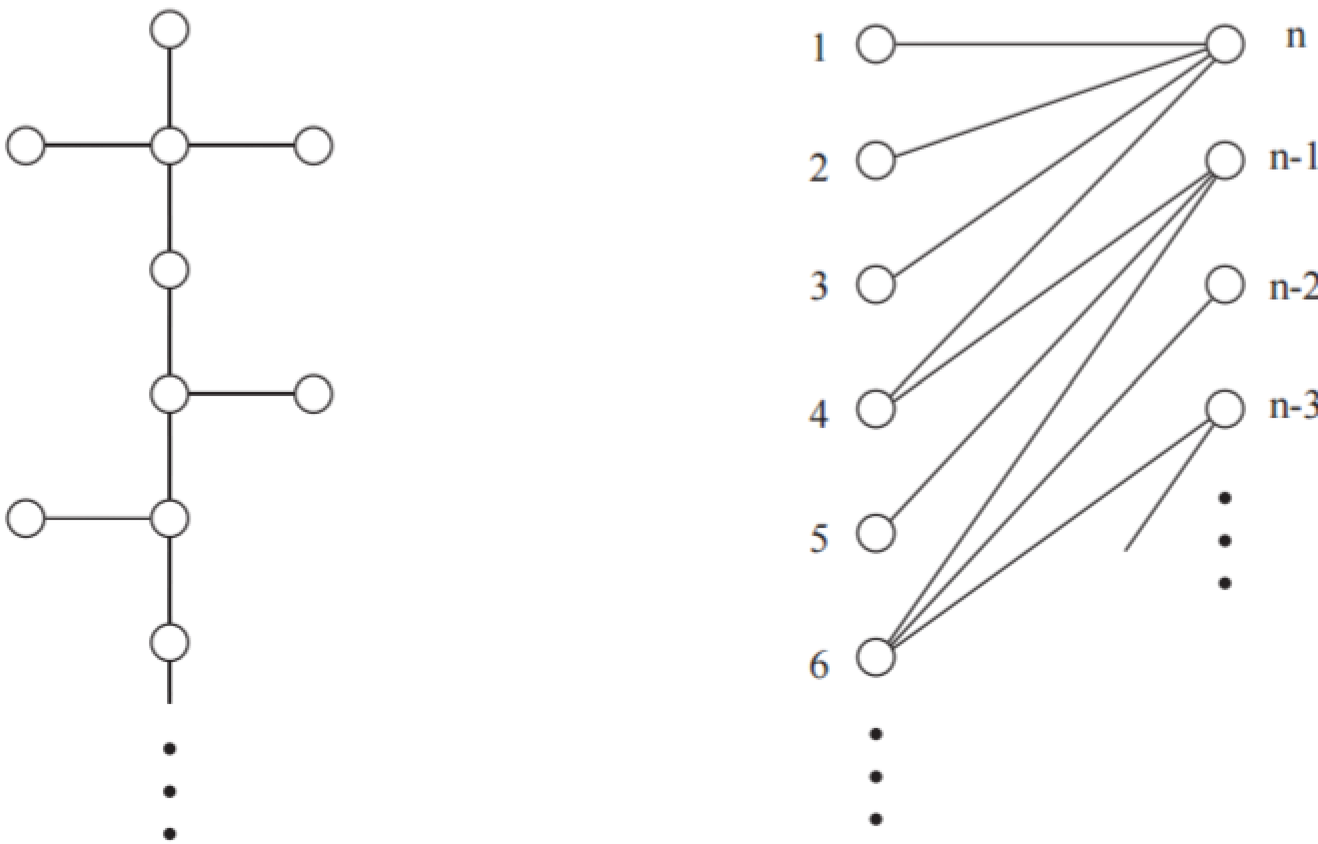
The resulting labeling gives labels n, n-1 ,..., n-$|X|$ + 1 to vertices in X while vertices of Y are labeled 1,2,3,...,$|Y|$.
Moreover , since $|X|$ + $|Y|$ = n, the labels of the vertices are all different.

Let l_i be the label of the vertex v_i . Clearly $l_1 < l_3 <$ ... and $l_0 > l_2 >$ ... .
Now for even i , if i $\notin \{0, k\}$, the neighbours of v_i have labels
$l_{i-1}, l_{i-1} + 1, ..., l_{i+1}$ and the induced edge labels are l_i -l_{i+1} , l_i -l_{i+1} -1 , ..., l_i -l_{i-1}.
For an odd i , if i$\neq$ k , the neighbour of the vertex v_i have
$l_{i-1}, l_{i-1} + 1, ..., l_{i+1}$ and the induced edge labels are l_{i-1} - l_i , l_{i+1} +1 - l_i, ..., l_i +1 - l_i .

The neighbours of v_0 have labels 1,2,...l_1 ,the neighbours of v_k have labels l_{k-1} , l_{k-1} +1 ,..., l_k -1 if k is even ,and l_k +1 ,l_k +2 ....,l_{k+1} if k is odd.
From all this it is easy to see that the n-1 edge labels are all different ,the largest is n-1 and the smallest is 1 .Thus the given labeling shown in the

figure is graceful.

FIGURE



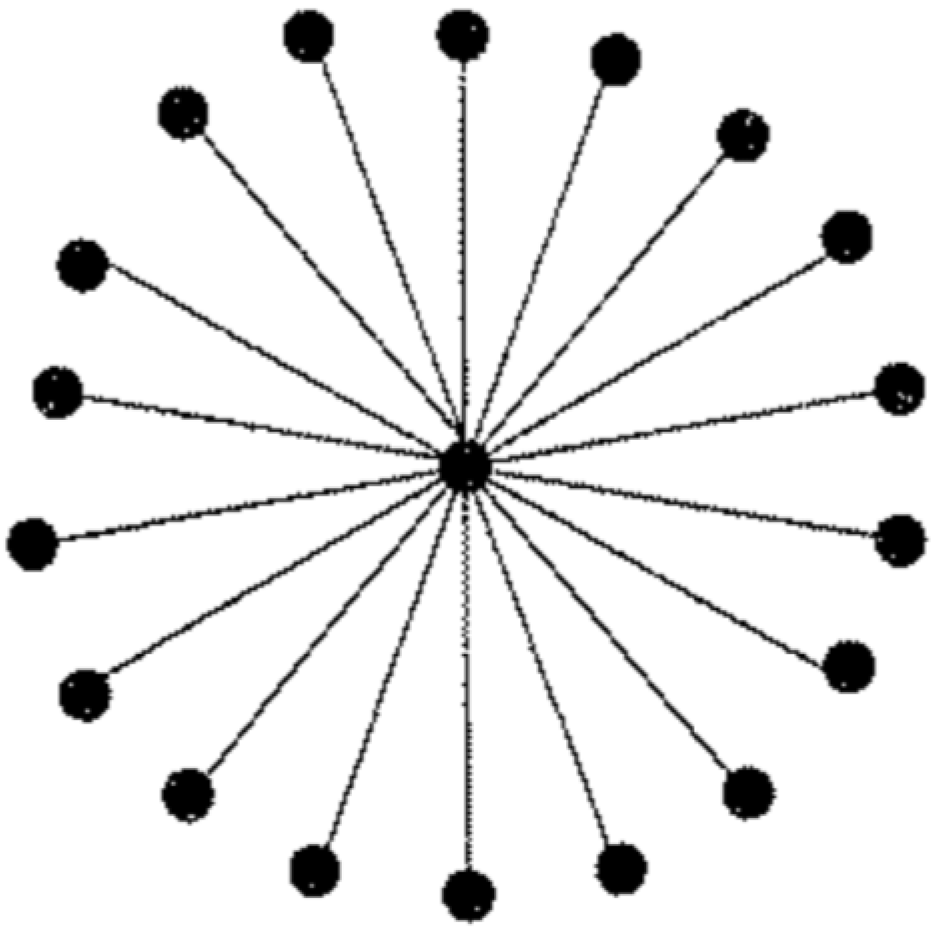
STAR

A star is any graph of the form $K_{1,k}$. The complete bipartite graph $K_{p,q}$ is a bipartite graph G =(A,B,E) such that $|A|$ =p , $|B|$ =q , and if u$\in$ A and v$\in$ B , then uv $\in$ E.In particular star graph is a complete bipartite graph $K_{1,q}$.

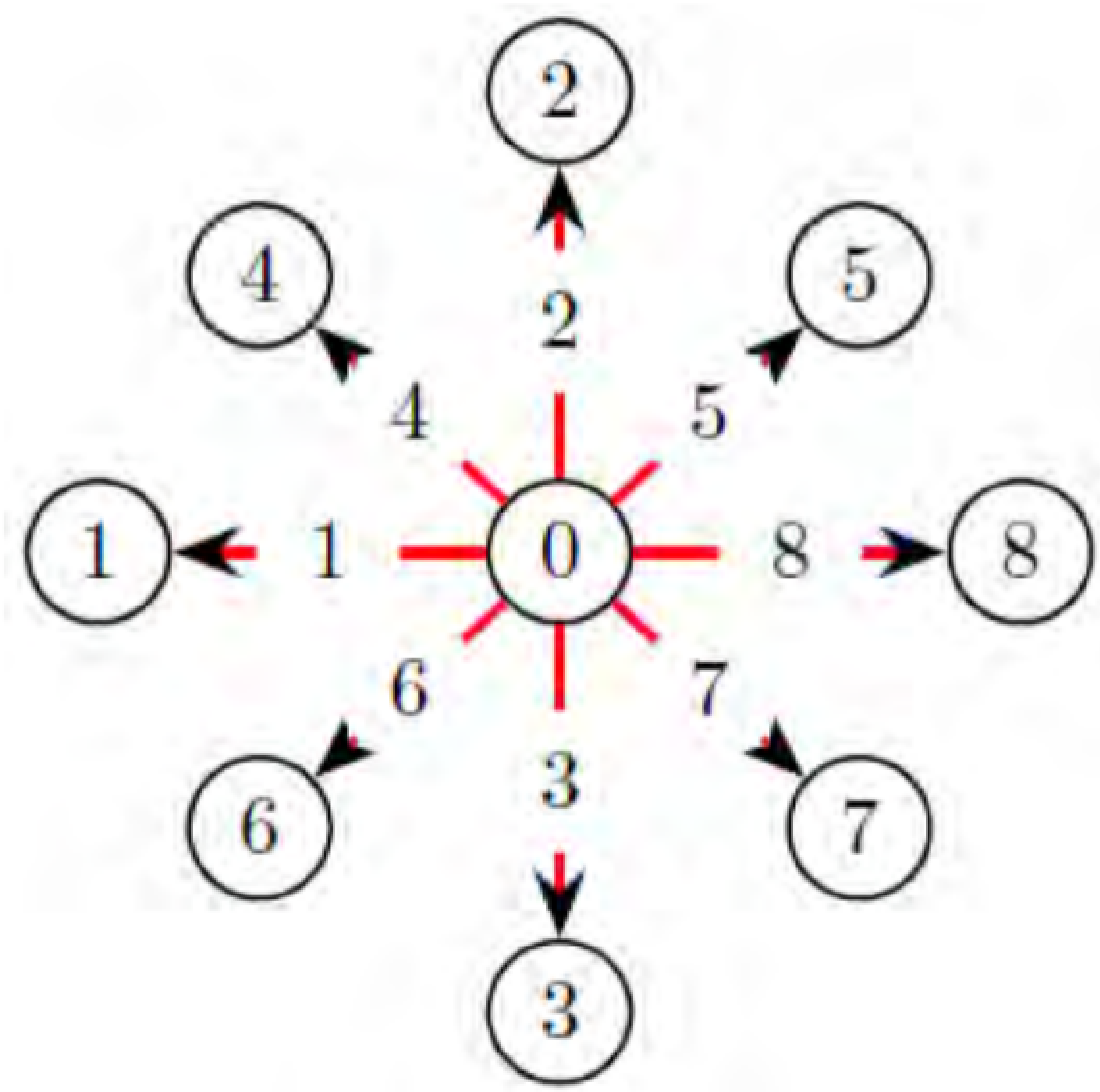
A tree with one internal node and k leaves is said to be a star $S_{1,k}$ that happen to be a complete bipartite graph $K_{1,k}$. Thus a star must be a tree on k+1 vertices with atleast k leaves.

Gven below is a figure of a star graph.

FIGURE



Stars can be gracefully labeled by labeling its centre by 0 and other leaves by j, j=1,2,3,...k.
An example of a gracefully labeled star is given below.



So stars are trees with one vertex that is adjacent to any number of leaves .It follows that paths and stars are simply caterpillars .That is a path is a caterpillar with exactly 2 leaves , while stars with atleast 2 leaves are caterpillars whose central paths have length 0.

Another early result of Golomb is that every tree on 5 vertices is graceful.He proves this by simply exhibiting a graceful labeling for all trees on 5 vertices.

Definition

Let G be a graph .Then the diameter of G , denoted by diam(G) is the length of a longest path in G .
We know show that any tree T with 1$\leq$ diam(T)$\leq$ 3 is a caterpillar .

THEOREM 7

Let T be a tree with diameter atleast 1 and atmost 3 .Then Tis a caterpillar.

PROOF

Assume that the diameter of the tree T is i , i$\in \{1, 2, 3\}$.
If i =1 , then clearly T = K_2 , we are done.
If i =2 , then let P =v_0, v_1, v_2 be a longest path in T.

Clearly v_0 and v_2 must be leaves otherwise P could be extended.
Thus if T has any other vertex y ,then there must be a path between y and v_1 that does not go through v_0 or v_2 .
We can extend this path by adding v_0 to its end.
Since 2 is the length of the longest path , this means that y must be attached to v_1 by an edge . Thus T is a star and we are done.

Fnally if i =3 ,let P = v_0, v_1, v_2, v_3 be a longest path in T. Clearly v_0 and v_3 must be leaves ,.Otherwise P could be extended. If T has any other vertex y, then there must be a path from y to a vertex of P that does not contain any other vertex of P.

This means we must have a path from y to v_1 or v_2 .Assume the path is to v_1 (the other case be handled similarly) .If the path from y to v_1 is not a single edge, then we obtain a new path that is of length atleast 4 by appending v_0 and v_2 . Thus ,any vertex y of T not on P is attached by a single edge to either v_1 or v_2 .
Therefore T is a caterpillar.

SPIDER TREES

A spider tree is a tree with atmost one vertex of degree greater than 2. If such a vertex exists , it is called the branch point of the tree. A leg of a spider tree is any one of the paths from the branch points to a leaf of the tree.
Also a spider is a tree consisting of m paths $P_{x_1}, P_{x_2}, ...P_{x_m}$ where the vertices $v_1, v_2, v_3..., v_m$,with each v_i is a leaf of P_{x_i} are identified .

Definition

Let G^1 =(V_1, E_1) and G^2 =(V_2, E_2) be graphs with no common vertices and let u$\in V_1$ and v$\in V_2$.
By identifying the vertex u with the vertex v we3 mean the procedure that results in the graph G=(V,E) where V =$(V_1 \cup V_2 - \{v\}$)
and E = $E_1 \cup (E_2 \backslash \{vx|vx \in E_2\}) \cup \{vx|vx \in E_2\}$

we will also call this the uv join of G^1 and G^2 and denote it by $G^1 \oplus G^2$
Now if G^1 and G^2 are trees , then clearly their uv join is also a tree.

LEMMA

Let f_1 be a bipartite labeling with labels starting at 0 of a tree S with f_2 (V) =0 for some v $\in$ V(T) .Then there is a graceful labeling of $S_u \oplus T_v$

THEOREM 8

The spider S $(x_1, x_2, x_3$) with 3 legs are graceful .

PROOF

The proof follows directly from the lemma identifying a leaf from the path P_{x_j} with a leaf of P_{x_2} .The resulting tree ,T is itself a path .Choose a graceful labeling of T such that the identified vertex has label 0.Then choose a bipartite labeling of P_{x_3} such that one of its leaves has label 0.Finally joining T and P_{x_3} and relabeling according to the lemma results in the tree S (X_1, X_2, X_3) which is labeled gracefully

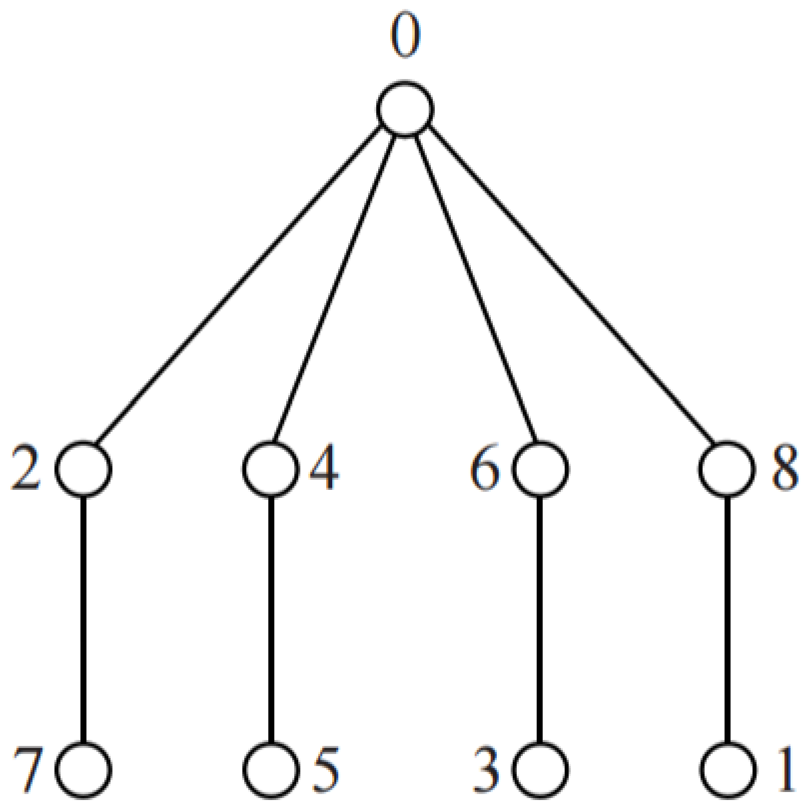
THEOREM 9

The spiders S (x_1, x_2, x_3, x_4) with 4 legs are graceful

PROOF

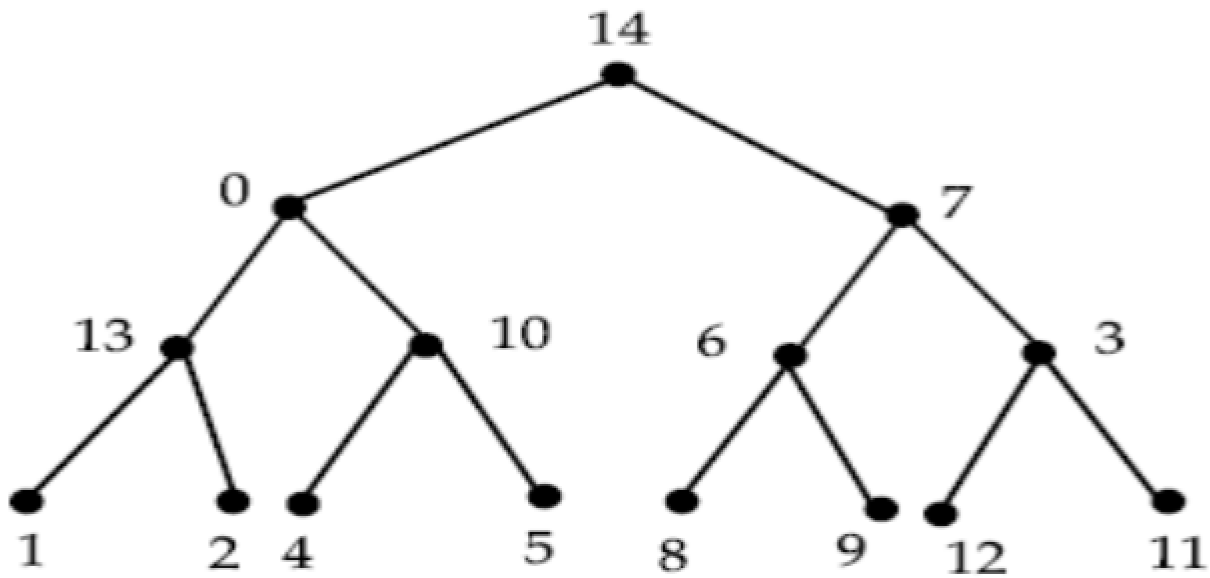
If at least one of x_1, x_2, x_3, x_4 is not 2, then without loss of generality $x_1 + x_2 \neq 4$.
Let u be the central vertex of the spider S $(x_1, x_2) = P_{x_1+x_2}$ and v be the centrel vertex of the spider S $(x_3, x_4) = P_{x_3+x_4}$. We know that there is a bipartite labeling of S (x_1, x_2) that labels u with 0 and a graceful labeling of S (x_3, x_4) that labels v with 0. The result follows from the lemma. The other possibility is that $x_1 = x_2 = x_3 = x_4$ =2.

The graceful labeling of S(2, 2 , 2, 2) as shown in figure below .



SYMMETRICAL TREES

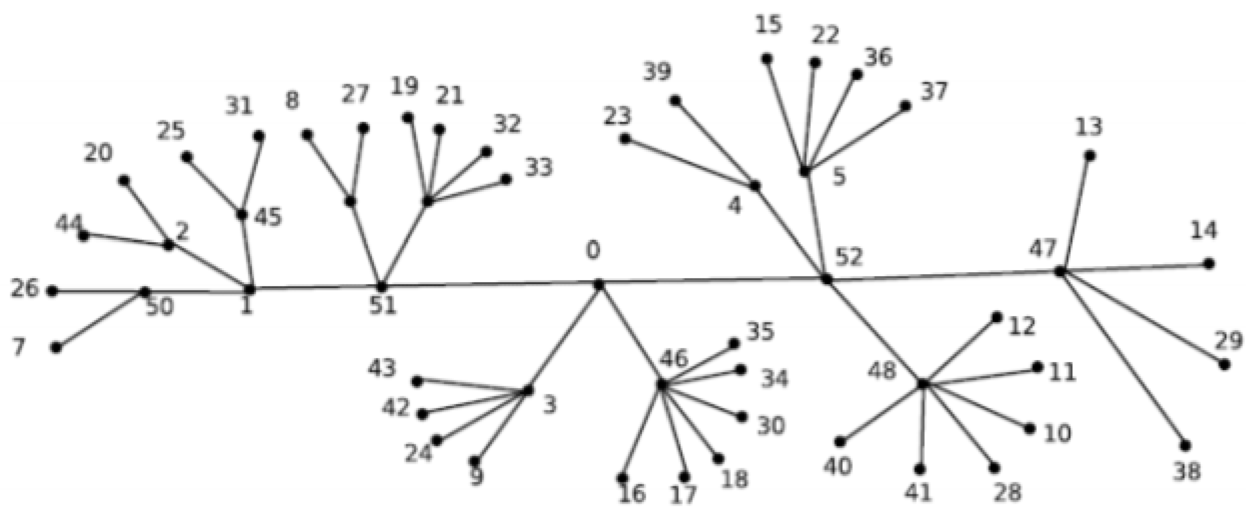
A rooted tree in which every level contains vertices of the same degree is called symmetrical trees.The following is a gracefully labeled symmetrical tree on 15 vertices.


14
0
7
13
10
6
3
1
2
4
5
8
9
12
11

Bermond and Schonheim proved that all symmetrical trees are graceful.

LOBSTER TREE

A lobster is a tree with the property that removal of all its leaves results in a caterpillar. Figure is illustrated as an example of graceful lobster tree.
In 1979 Bermond conjectured that lobsters are graceful.
In 2002, that Morgan proved that all lobster trees with perfect matching are graceful.
Mishra and Panigrahi found classes of graceful lobster of diameter atleast 5.
They also showed that some other classes of lobsters are graceful.

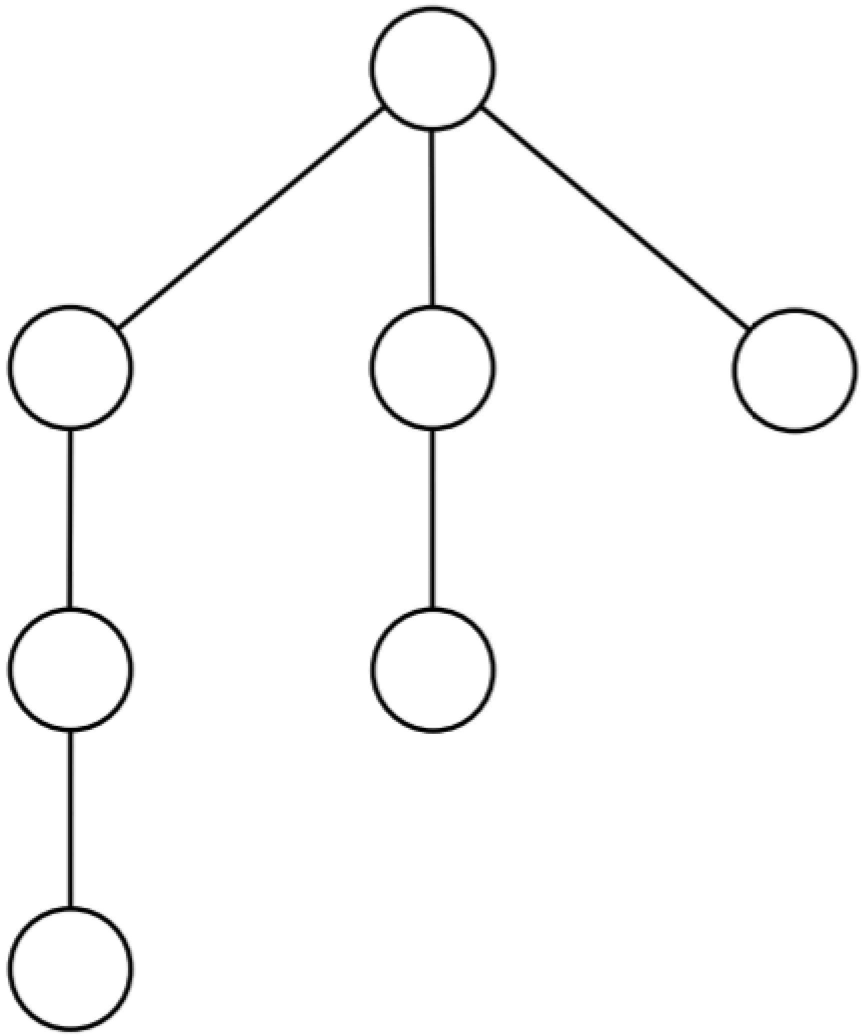


TREES WITH DIAMETER 5

The diameter of a graph or a tree is the maximum of the shortest paths between its vertices.
Hrnciar and Harier extended the proof for caterpillars to show that all trees with diameter ≤ 5 are graceful.

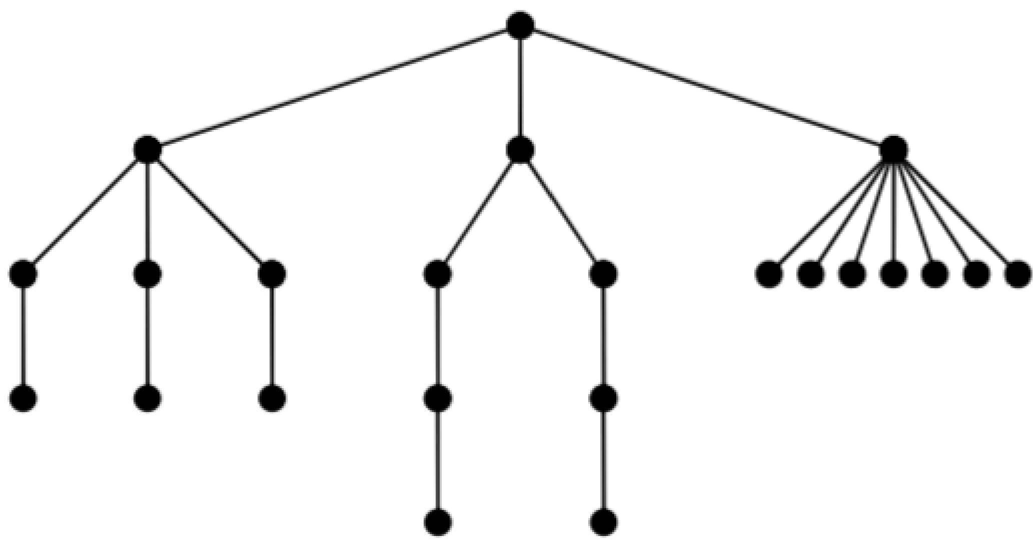
OLIVE TREES

An olive tree has a root node with k branches attached ; the ith branch has length i.
Pastel and Raynaud proved that all olive trees are graceful.



BANANA TREES

A banana tree is constructed by bringing mutiple stars together at a single vertex. Banana trees have not been proved graceful, although Bhat -Nayak and Deshmukh have proven the gracefulness of certain classes of banana tree.



SOME USEFUL RESULTS

- The cycle graph C_p iff 4| p or 4 | p+1.
- If G is a (p,q) graceful graph , then 4 | or 4 | (q +1) .
- K_2 , K_3 , K_3 are the only graceful graphs.

- The complete bipartite graph $K_{p,q}$ is graceful for all p,q$\geq$ 1

PROOF

Let G =(A,B,E) be a bipartite graph with a=$|A|$ and b=$|B|$.
Assign the vertices from A with numbers 0,1,2,...a-1, and assign the vertices from B with numbers a, 2a, ..., ba.

- K_n is graceful iff n$\leq$ 4.

- $K_{m,n}$ is graceful.

- $K_{1,m,n}$ is graceful.

- $K_{1,1,m,n}$ is graceful.

CONSTRAINT SATISFACTION PROBLEM

Around the life span of the conjecture ,there has been some research done on showing it with a particular algorithm or with the help of constraint programming .

At first Alred et .al proved that trees upto 27 vertices are graceful. This result was based on a single algorithm they provided and was later extended by Horton to 29 vertices with his edge search algorithm.

But the most promising result was by Fang where he proved that any tree with 35 vertices were graceful with the help of deterministic and hill climbing search technique.

He proposed a hybrid algorithm consisting deterministic back tracking and probablistic hill climbing for any random tree with 35 vertices.
Even with this algorithm ,the project is likely to take 7.7 years according to the statistics. But as the computational power is getting higher each day , a promising distributed method can make a better result out of this approach.

CHAPTER 4

APPLICATIONS OF GRACEFUL TREES

The question that arises with this unsolved conjecture is that " What is the point behind graceful ? Though much work has been done on the topic to find the graceful labels , minimal effort has been given to find its applications. We present some of those ideas along with a new one of ours on the applications of gracefully labeled trees and graphs.

Graph labeling were introduced in the mid sixties. In the middle of the time dozens of the graph lebeling technique have been introduced over 1000 papers. An enormous emount of literature has grown around the subject and is still getting flourished due to increasing number of application oriented concepts.Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications.

In the diverse fields of human enquiry a qualitative labeling of graph labeling of graph elements have inspired research such as conflict resolution in social psychology ,electrical circuit theory and energy crisis etc and quantitative labeling of graphs have led to quite intricate fields of application such as coding theory problems including the design of good radar location codes; synch- set codes; missile guidance codes and convolution codes with optimal autocorrelation properties.

Moreover labeled graphs have also applied in determining ambiguities in X-Ray crystallogrphic analysis to design communication network addresing systems,in determining optimal circuit layouts and Radio- astronomy etc.

The Coding Theory

The design of good non periodic codes for pulse radar and missile guidance is equivalent to labeling the complete graph in such a way that all the edge labels are distinct. therefore node labels can determine the time positions at which pulses are transmitted.

Xray crystallography

In modern days Xray diffraction is one of the most powerful technique for characterizing the structural properties of crystalline solids in which a beam of X rays strikes a crystal and diffracts into many specific directions.Position of atom in a crystal structures are made by X ray diffraction patterns.Measurements indicate the set of inter atomic distances in crystal lattices. Mathematically , one can find the finite set of integers to one atom position ,so that diffraction is equivalent to the distinct edge length between these two integers . In some cases more than one structure has the same diffraction information .This problem is mathematically equivalent to determining all labeling of the appropriate graphs which produce a pre specified set of edge labels.

Network Addressing

The most anticipated and perhaps the best use of graceful labeling is in network addressing .In a wireless network , each station is assigned a channel (a positive integer) such that inteference can be avoided. With the smaller distance ,the inteference between the stations become higher ,so the difference in channel assignment has to be greater.For this assignment graceful labeling becomes handy as they are unique for the the whole system.

Graceful labeling on graph data bases

In the era of big data , graph databases are getting popular day by day.Every social or relational data can be stored in a graph data base.These databases will store millions of relations between users and their behaviour .So querying in these databases need to be faster with the help of indexing.But in these format database we cannot really put up a good indexing technique as it will slow down the data insertion operation quite a bit. So we introduce gracefully labeled vertices as the index in these data.

We can generate a graceful label and we can use it as the key for indexing the data base engine .So a query for that particular data will be much faster as we have a unique key for it throughout the whole tree or graph.As the graceful labeling algorithm will tell us about the neighbours of the queried vertex. wWe can tell the surrounding node's position with this too.

Graceful labeling technique is a most popular graph labeling technique and traces its origin to one introduced by Rosa. However graceful labeling is not just an appealing research problem that a non researcher or non specialist can understand but a problem with extra ordinary versatile applications.

Graceful labeling of trees have been used in Multi Protocol Label Switching (MPLS) routing platform in IP networks.The MPLS technique has many useful application in networking . For example it can be effectively used in ATM network infrastructure, as the labeled flows can be mapped to ATM virtual circuit identifiers and vice versa.

Graceful graphs are easy to understand, but have many seemingly simple unresolved questions that no one has been able to answer yet.Graceful graphs lend themselves naturally to computational approaches. Graceful graphs have wider scope for implementation mainly in the field of networking and with respect to MPLS protocol.

Graceful labeling of directed graphs have been used to describe some algebraic structures such as cyclic difference sets , sequenceable groups ,generalised complete mappings and neo fields.The advances in graceful labeling would also have an impact on many other areas.

CONCLUSION

Graceful labeling of trees has been marked as a disease from mathematicians to scientists who come in touch with it. There are multiple methods existing to solve a specific structured tree but not a single algorithm or method has been devised to answer the conjecture.

We can hope with the modern age technology and faster computational powers , we will come up with a certain result on this year old mistery.

In chapter 1 we discuss about the preliminaries.
In chapter 2 , we introduce the problem of graceful labeling.
Chapter 3 deals with the different classes of trees that have been proved graceful to strengthen the Graceful tree conjecture that "All trees are graceful.
Chapter 4 contains application of graceful labeling in the real world problem.

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