# FIXED POINT THEORY

***A Dissertation submitted in partial fulfillment of***

***The***

***Requirement for the award of***

**DEGREE OF MASTER OF SCIENCE**

**IN MATHEMATICS**

***By***


**RIYANA SREEDHARAN**

**REGISTER NO: SM16MAT013**


**(2016 – 2018)**


**DEPARTMENT OF MATHEMATICS**

**ST.TERESA'S COLLEGE, (AUTONOMOUS)**

**ERNAKULAM, KOCHI - 682011**


**APRIL 2018**

# DEPARTMENT OF MATHEMATICS

# ST.TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM

## CERTIFICATE

This is to certify that the dissertation entitled "FIXED POINT THEORY" is a bonafide record of the work done by RIYANA SREEDHARAN under my guidance as partial fulfillment of the award of the degree of Master of Science in Mathematics at St. Teresa's College (Autonomous), Ernakulam affiliated to Mahatma Gandhi University, Kottayam. No part of this work has been submitted for any other degree elsewhere.

**Miss. Anna Treesa Raj (Supervisor)**

Assistant Professor

Department Of Mathematics

St.Teresa's College (Autonomous)

Ernakulam

**Smt. Teresa Felitia (HOD)**

Associate Professor

Department of Mathematics

St. Teresa's College (Autonomous)

Ernakulam

**External Examiners:**

**1. ..................................................**

**2. ...............................................**

# DECLARATION

I hereby declare that the work presented in this project is based on the original work done by me under the guidance of Miss. Anna Treesa Raj, Assistant Professor, Department of Mathematics, St Teresa's College(Autonomous), Ernakulam and has not been included in any other project submitted previously for the award of any degree.

ERNAKULAM RIYANA SREEDHARAN

APRIL 2018 **REGISTER NO: SM16MATO13**

# ACKNOWLEDGEMENT

I express my sincere gratitude to Miss. Anna Treesa Raj, Assistant Professor, Department of Mathematics, St. Teresa's College (Autonomous), Ernakulam for helping me in the successful completion of this work.

I further express my gratitude to Smt. Teresa Felitia, Head of the Department of Mathematics, St. Teresa's College (Autonomous), Ernakulam and other teachers of the department for their encouragement, suggestions and assistance in taking up this dissertion.

I finally thank my parents, friends and all well wishers who supported me during this preparation of this project. I bow my head before God almighty, without whose blessings, I wouldn't be able to complete this work.

ERNAKULAM

APRIL 2018

RIYANA SREEDHARAN

**REGISTER NO: SM16MAT013**

# Contents

# Chapter 1

# INTRODUCTION

Mathematics is one of the essential emanation of the human spirit-a thing to be valued in and for itself like art or poetry.That is, mathematics may not teach us how to add love or minus hate,but it gives us every reason to hope that every problem has a solution.

The theory of fixed point is one of the most powerful tool of modern mathematics.Theorem concerning the existence and properties of fixed points are known as fixed point theorem.Fixed point theory is a beautiful mixture of analysis,topology and geometry.In particular fixed point theorem has been applied in such field as mathematical engineering,physics,economics,biology,game theory and chemistry etc.

The fixed point theory is divided into two major areas: One is the fixed point theory on contraction or contraction type mappings on complete metric space and the second one is the fixed point theory on continuous operators on compact and convex subsets of a normed space.The begining of fixed point theory in normed space is attributed to the work of Brouwer in 1920, who proved that any continuous self-map of the closed unit ball of $R^n$ has a fixed point.Thebegining of fixed point theory on complete metric space is related to Banach Contraction Principle,in 1922.Let (X, d) be a metric space and F : X→X be a mapping.Then F is said to be a contraction if there exist a constant L ∈[0,1) ,called a contraction factor,such that d(F(x), F(y))≤ Ld(x, y) Banach contraction principle says that any contraction self-mappings on a complete metric space has a unique fixed point.This principle is one of a very power test for existence and uniqueness of the solution of considerable problems arising in mathematics.

The aim of this paper is to surveys the basic theorem for contraction mappings on a finite dimensional metric space.This work consists of mainly eight Chapters.The first two-three chapters concerning the history and some basic definitions which are useful for this work. In the fourth chapter,we deals with the existence and uniqueness of fixed points for a contractive mapping in a complete metric space.A fixed point of an operator is a solution of the equation x = F(x). Concerning the metric branch,the most important metric fixed point result is the Banach Contraction Principle which was considered as one of the fundamental principle in the field of functional analysis.Then we establish the extension of Banach Contraction Principle for weak contraction mapping and the sequences of mapping and uniqueness of fixed point.In the last two chapters,we treat fixed points in a b-metric space for weak contraction mapping and in compact metric space by introducing the new concept co-cyclic representation.

# Chapter 2

# HISTORY

In 1886,Poincare was the first to work in this field.Then Brouwer in 1912,proved fixed point theorem for the solution of the equation F(x) = x. Brouwer theorem gives no information about the location of fixed points. However, effective methods have been developed to approximate the fixed points. He also proved fixed point theorem for a square,a sphere and their n-dimensional counter parts which was further extended by Kakutani.Mean while Banach Contraction Principle came into existence which was considered as one of the fundamental principle interested the field of functional analysis.In 1922,Banach proved that a contraction mapping in the field of a complete metric space possesses a unique fixed point.Later on it was developed by Kannan.The first fixed point theorem in an infinite dimensional Banach space was given by Schauder in 1930. Fixed point theory is an interdisciplinary topic which can be applied in various disciplines of mathematics and mathematical science like game theory,approximation theory, economics,optimization theory and variational inequalities.The fixed point theory (as well as Banach Contraction Principle) has been studied and generalized in different spaces and various fixed point theorem were developed.

# Chapter 3

# PRELIMINARIES

A point is often called fixed point when it remains invarient,irrespective of the type of transformation it undergoes.For a functionF that has a set X as both domain and range,a fixed point is a point x ∈ X for which F(x) = x.Two fundamental theorem concerning fixed points are those of Banach and Brouwer.

**Definition 3.1.** *Let X be a non-empty set and F be a function which maps X to X.A fixed point of F is a point $x \in X$ such that $F(x) = x$*

**Example 1.** *Let F be a map on R defined by $F(x) = x\ 2\ \ 7x + 12$ We know that x=3,4 are root of the equation. Consider $F(x) = x$ where $F(x) = \frac{x^2+12}{7}$*

*Then $x = 3$ and $x = 4$ are two fixed point of $F(x)$*

**Definition 3.2.** *Let X be a non-empty set.A mapping $d : X\ \ X \to R$ is said to be a metric(or distance function) iff d satiesfies the following axioms: (M-1) $d(x, y) \geq 0\ \forall\ x, y \in X$*

*(M-2) $d(x, y) = 0$ iff $x = y\ \forall\ x, y \in X$*

*(M-3) $d(x, y) = d(y, x)\ \forall\ x, y \in X$*

*(M-4) $d(x, y) \leq d(x, z) + d(z, y)\ \forall\ x, y, z \in X$*

If d is metric for x,then the ordered pair(X, d) is called a metric space and d(x, y) is called the distance between x and y

**Definition 3.3.** *A sequence* $(x_n)$ *in a metric space* $(E, d)$ *is said to converge to an element* $x of E$ *if*

$$\lim_{n\to\infty} d(x_n, x) = 0$$

*A sequence* $x_n$ *of elements of a metric space* $(E, d)$ *is called a Cauchy sequence if given* $\epsilon > 0$*, there exists* $N$ *such that for* $p, q \geq N$,$d(x_p, x_q) < \epsilon$

**Definition 3.4.** *A metric space (X, d) is called complete,if every cauchy sequence converges in it.*

**Definition 3.5.** *Let (X, d) be a complete metric space and let F:X* $\rightarrow$ *X is said to be Lipschitz constant such that d(F(x), F(y))* $\leq$ *c d(x, y)* $\forall$ *x, y*$\in$*X*

*1. If o¡c¡1,then the mapping is called contraction*

*2. If c=1,then the mapping is called nonexpansive*

*3. If d(F(x), F(y)) = d(x, y),then it is isometry*

**Example 2.** *Let F:R*$\rightarrow$*R by F(x)=*$\frac{x}{2}$ *d(F(x), F(y))leq* $\frac{1}{2}$ *d(x, y) here c=* $\frac{1}{2}$ *,0¡c¡1. Hence F is a contraction.*

**Example 3.** *Let F:[* $\frac{1}{2}$*,2]*$\rightarrow$*[*$\frac{1}{2}$*,2] by F(x) = 1 x d(F(x), F(y)) = d(1 x.1 y)* $\leq$ *d(x, y) Hence F is nonexpansive.*

**Example 4.** *Let F:R* $\rightarrow$ *R by F(x) = x Then d(F(x), F(y)) = d(x, y) Hence F is an isometry.*

**Definition 3.6.** *A subset A of a metric space X is said to be c compact subset of X ,if every cover of A by open subsets of X has a finite subcover. A metric space X is said to be compact if X is a compact subset of itself.In other words ,every sequence in X has a convergent subsequence.*

**Definition 3.7.** *A neighborhood of* $x_0 \in X$ *is an open ball of radius* $r > 0$*in* $X$ *that is centered at* $x_0$*. An open ball of radius* $r$ *centered at* $x0$ *is the collection of all points* $x \in X$ *satisfying* $|x\text{-}x_0| < r$

**Definition 3.8.** *A set is said to be locally compact if every point in* $X$ *has a compact neighborhood .That is , for every point in* $x \in X$ *we can find an open ball for which every sequence has a convergent subsequence*

**Definition 3.9.** *A mapping*$T$ *of a metric space* $E$ *into a metric space* $E$ *is said to be continuous if for every convergent sequence* $(x_n)$ *of* $E$*,*

$$\lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n)$$

# Chapter 4

# FIXED POINT THEOREMS

Fixed point theorems concern maps $f$ of a set $X$ into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology.

### 4.0.1 SOME OF THE INITIAL THEOREMS

Some of the basic theorems in this subject is listed below without proof.

**Theorem 4.1** ( Brouwer fixed point theorem )**.** *The theorem states that if $f : B \to B$is a continuous function and $B$ is a ball in $R^n$ , then $f$ has a fixed point.*

This theorem simply guarantees the existence of a solution, but gives no information about the uniqueness and determination of the solution. For example,

**Example 5.** *if $f : [0,1] \to [0,1]$is given by $f(x) = x^2$ , then $f(0) = 0 and f(1) = 1$, that is, $f$ has 2 fixed points.*

Several proofs of this theorem are given. Most of them are of topological in nature. A classical proof due to Birkhoff and Kellog was given in 1922. Brouwer theorem gives no information about the location of fixed points.

However, effective methods have been developed to approximate the fixed points. Such tools are useful in calculating zeros of functions REMARK: *This theorem is not true in infinite dimensional spaces.*
The first fixed point theorem in an infinite dimensional Banach space was given by Schauder in 1930.

**Theorem 4.2** (SCHAUDER FIXED POINT THEOREM). *If $B$ is a compact, convex subset of a Banach space $X$ and $f : B \to B$ is a continuous function, then f has a fixed point*

The Schauder fixed point theorem has applications in approximation theory, game theory and other scientific area like engineering, economics and optimization theory. The compactness condition on B is a very strong one and most of the problems in analysis do not have compact setting. It is natural to prove the theorem by relaxing the condition of compactness. Schauder proved the following theorem

**Theorem 4.3.** *If $B$ is a closed bounded convex subset of a Banach space $X$ and $f : B \to B$ is continuous map such that f(B) is compact, then f has a fixed point.*

The above theorem was generalized to locally convex topological vector spaces by Tychonoff in 1935 .

**Theorem 4.4.** *If $B$ is a nonempty compact convex subset of a locally convex topological vector space $X$ and $f : B \to B$ is a continuous map, then f has a fixed point.*

Further extension of Tychonoffs theorem was given by Ky Fan . A very interesting useful result in fixed point theory is due to Banach known as the Banach contraction principle.

**Theorem 4.5.** *Every contraction map is a continuous map, but a continuous map need not be a contraction map.*

**Example 6.** *$fx = x$is a continuous map but it is not a contraction map.*

The method of successive approximation introduced by Liouville in 1837 and systematically developed by Picard in 1890 culminated in formulation by Banach known as the Banach contraction principle (BCP) is stated in the following section.

### 4.0.2 BANACH CONTRACTION PRINCIPLE

**Theorem 4.6** (BANACH CONTRACTION PRINCIPLE OR CONTRACTION THEOREM)**.** *Let $F$ be a contraction on a complete metric space $X$.Then $F$ has a unique fixed point.*

*Proof.* Let $x_0 \in X$ Define $x_1 = F(x_0)$ We will show that the sequence $(x_n)$ is cauchy sequence in $X$. Since $F$ is a contraction,there exist a real number $0 < c < 1$ such that,

$$d(F(x), F(y)) \leq cd(x, y)\forall x, y \in X$$

For $n \geq 1$,

$$d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1}))$$

$$\leq cd(x_n, x_{n-1})$$

Similarly,

$$d(x_n, x_{n-1}) = d(F(x_{n-1}), F(x_{n-2}))$$

$$\leq cd(x_{n-1}, x_{n-2})$$

Thus

$$d(x_{n+1}, x_n) \leq c2d(x_{n-1}, x_{n-2})$$

Continuing like this,we get,

$$d(x_{n+1}, x_n) \leq c^n d(x, x_0) \forall n = 0, 1, 2, ...$$

Now for $m > n$,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ..... + d(x_{m-1}, x_m)$$

$$\leq (c^n + c^n + 1 + \cdots + c^{m-1})d(x_1, x_0)$$

$$\leq c^n(1 + c + ..... + c^{m-n-1})d(x_1, x_0)$$

$$\leq c^n d(x_1, x_0)\frac{1}{1-c}$$

$$= c^n d(F(x_0), x_0)\frac{1}{1-c}$$

Since $c^n \to \infty asn \to \infty$ $(x_n)$ is a cauchy sequence in $X$. Sine $X$ is complete $\lim_{n\to\infty(x_n)} = x$ for some $x \in X$ Since $F$ is a contraction,$F$is continuous and hence

$$F(x) = F(\lim_{n\to\infty} x_n)$$

$$= \lim_{n\to\infty} F(x_n)$$

$$= \lim_{n\to\infty} x_{n+1}$$

$$= x$$

Thus suppose there exist $y \in X$ such that$F(y) = y$. Then ,

$$d(F(x), F(y)) \leq d(x, y)$$

$$d(x, y) \leq cd(x, y)$$

$$d(x, y) = 0$$

$$\Rightarrow x = y$$

Thus,$F$ has a unique fixed point in $X$

□

**Example 7.** *Let $F : R \to R$ defined by, $F(x) = 3x - 1$ Then $x = \frac{1}{2}$ is the fixed point of $F$.*

REMARK:

*Letting $m \to \infty$ in the above equation, $d(x_n, x) \leq c^n d(F(x_0), x_0)\frac{1}{(1-c)}$ which provides a control on the convergence of $(x_n)$ to the fixed point $x$.*

REMARK:

*Contraction on incomplete metric space may fail to have fixed points.*

**Example 8.** *Let $X = (0, 1] with usual distance. Define F : X \to X as F(x) = \frac{x}{2}$*

*Then*

$$\begin{aligned} d(F(x), F(y)) &= d(\frac{x}{2}, \frac{y}{2}) \\ &\leq \frac{d(x, y)}{2} \end{aligned}$$

*which implies that F is a contraction on X. Now, X is an incomplete space,since $(xn) = (\frac{1}{n})$ is a cauchy sequence which converges to zero,but $0 \notin X$.*

**Corollary 4.6.1.** *Let $X$ be a complete metric space and $Y$ be a topological space. Let $f : X \times Y \to X$ be a continuous function. Assume that $f$ is a contraction on $X$ uniformly in $Y$ ,that is,*

$$d(f(x_1, y), f(x_2, y)) \leq \lambda d(x_1, x_2),$$

*$\forall x_1, x_2 \in X, \forall y \in Y for some \lambda < 1$. Then, for every fixed $y \in Y$ , the map $x \mapsto f(x, y)$ has a unique fixed point $\phi(y)$.*
*Moreover, the function $y \mapsto \phi(y)$is continuous from $Y to X$.*

Notice that if $f : X \times Y \to X$ is continuous on $Y$ and is a contraction on $X$ uniformly in $Y$ , then $f$ is in fact continuous on $X \times Y$ .

*Proof.* In light of Theorem 4.1, we only have to prove the continuity of $\phi$.

For $y, y_0 \in Y$, we have $d(\phi(y), \phi(y_0)) = d(f(\phi(y), y), f(\phi(y_0), y_0))$

$$\leq d(f(\phi(y), y), f(\phi(y_0), y)) + d(f(\phi(y_0), y), f(\phi(y_0), y_0))$$

$$\leq \lambda d(\phi(y), \phi(y_0)) + d(f(\phi(y_0), y), f(\phi(y_0), y_0))$$

which implies

$$d(\phi(y), \phi(y_0)) \leq \frac{1}{1-\lambda} d(f(\phi(y_0), y), f(\phi(y_0), y_0)).$$

Since the above right-hand side goes to zero as $y \to y_0$, we have the desired continuity ☐

**Example 9.** *Consider the map*

$$g(x) = \begin{cases} \frac{1}{2} + 2x & x \in [0, \frac{1}{4}] \\ \frac{1}{2} & x \in [\frac{1}{4}, 1] \end{cases}$$

*mapping* $[0, 1]$ *onto itself.*

*Then* $g$ *has unique fixed point(*$x = \frac{1}{2}$*)*

*For*

$g(x) = x \Rightarrow x = \frac{1}{2} + 2x$

$\Rightarrow x = \frac{-1}{2} \notin [0, 1]$

$g(x) = x \Rightarrow x = \frac{1}{2} \in [0, 1]$ *.And* $g$ *is discontinous.*

**Definition 4.1.** *For* $f : X \to X$ *and* $n \in N$*, we denote by* $f^n$ *the* $n^{th}$ *-iterate of f, namely,*$f \circ \cdots \circ f$ *n-times (*$f^0$ *is the identity map)*

**Corollary 4.6.2.** *Let* $X$ *be a complete metric space and let* $f : X \to X$*. If* $f^n$ *is a contraction, for some* $n \geq 1$*, then* $f$ *has a unique fixed point* $x \in X$*.*

*Proof.* Let $x$ be the unique fixed point of $f^n$ , given by Theorem 4.1. Then $f^n(f(x)) = f(f^n(x)) = f(x)$, which implies $f(x) = x$. Since a fixed point of f is clearly a fixed point of $f^n$ , we have uniqueness as well. ☐

**Example 10.** *.In the abovr example we can see that* $g^2(x) \equiv \frac{1}{2}$.

# Chapter 5

# EXTENSIONS OF CONTRACTION PRINCIPLE

There is in the literature a great number of generalizations of Theorem 4.6

**Theorem 5.1** (*Boyd-Wong*)**.** *Let $X$ be a complete metric space, and let $f$ : $X \to X$. Assume there exists a right-continuous function phi : $[0, \infty) \to [0, \infty)$such that $\phi(r) < rifr > 0$,and $d(f(x_1), f(x_2)) \leq \phi(d(x_1, x_2)), \forall x_1, x_2 \in X$. Then $f$ has a unique fixed point $x \in X$. Moreover, for any $x_0 \in X$ the sequence $f^n(x_0)$ converges to $x$.*

Clearly, Theorem 4.6 is a particular case of this result, for $\phi(r) = \lambda r$.

*Proof.* If $x_1, x_2 \in X$are fixed points of $f$, then

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \phi(d(x_1, x_2)$$

so $x_1 = x_2$. To prove the existence, fix any $x_0 \in X$, and define the iterate sequence $x_n + 1 = f(x_n)$. We show that $x_n$ is a Cauchy sequence, and the desired conclusion follows arguing like in the proof of Theorem 4.6. For $n \geq 1$, define the positive sequence $a_n = d(x - n, x_{n-1})$ It is clear that $a_{n+1} \leq \phi(a_n) \leq a_n$; therefore an converges monotonically to come $a \geq 0$. From the right-continuity of $\phi$, well get $a \leq \phi(a)$, which entails $a = 0$. If $x_n$ is not a Cauchy sequence, there is $\epsilon > 0$and integers $mk > nk \geq k$ for every

$k \geq 1$ such that

$$d_k := d(x_{m_k}, x_{n-k}) \geq \epsilon, \ \forall k \geq 1.$$

In addition, upon choosing the smallest possible $m_k$, we may assume that

$$d(x_{m_{k-1}}, x_{n_k}) < \epsilon$$

for k big enough (here we use the fact that an $\to$0). Therefore, for k big enough, $\epsilon \leq d_k \leq d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_{k-1}, x_{n_k}}) < a_{m_k} + \epsilon$ implying that $d_k \to \epsilon$ from above as $k \to \infty$. Moreover,

$$d_k \leq d_{k+1} + a_{m_{k+1}} + a_{n_{k+1}} \leq \phi(d_k) + a_{m_{k+1}} + a_{n_{k+1}}$$

and taking the limit as $k \to \infty$ ,we obtain the relation $\epsilon \leq \phi(\epsilon)$ which has to be false since $\epsilon > 0$. □

**Theorem 5.2** (*Caristi*). *Let $X$ be a complete metric space, and let $f : X \to X$. Assume there exists a lower semicontinuous function $\psi : X \to [0, \infty)$such that*

$$d(x, f(x)) \leq \psi(x)\psi(f(x)), \forall x \in X.$$

*Then $f$ has (at least) a fixed point in $X$.*

Again, Theorem 4.6 is a particular case, obtained for,

$$\psi(x) = d(x, f(x))/(1 - \lambda).$$

Notice that f need not be continuous.

*Proof.* We introduce a partial ordering on X, setting

$$x \preceq y \Leftrightarrow d(x, y) \leq \psi(x) - \psi(y).$$

Let $\emptyset \neq X_0 \subset X$ be totally ordered, and consider a sequence $x_n \in X_0$such that $\psi(x_n)$ is decreasing to $\alpha := inf \psi(x) : x \in X_0$. If $n \in N$ and $m \geq 1$,

$$d(x_{n+m}, x_n) \leq \sigma_{i=0}^{m-1} d(x_{n+i+1}, x_{n+i})$$

$$\leq \sigma_{i=0}^{m-1}\psi(x_{n+i})\psi(x_{n+i+1})$$

$$= \psi(x_n)\psi(x_{n+m}).$$

Hence $x_n$is a Cauchy sequence, and admits a limit $x \in X$, for X is complete. Since $\psi$ can only jump downwards (being lower semicontinuous), we also have $\psi(x) = \alpha$. If $x \in X_0$ and $d(x, x) > 0$, then it must be $x \preceq x_n$ for large n. Indeed, $\lim_n \psi(x_n) = \psi(x) \leq \psi(x)$. We conclude that $x$ is an upper bound for $X_0$, and by the Zorn lemma there exists a maximal element x. On the other hand, $x \preceq f(x)$, thus the maximality of x forces the equality $x = f(x)$. □

If we assume the continuity of f, we obtain a slightly stronger result, even relaxing the continuity hypothesis on $\psi$.

**Theorem 5.3.** *Let X be a complete metric space, and let $f : X \to X$ be a continuous map. Assume there exists a function $\psi : X \to [0, \infty)$ such that*

$$d(x, f(x)) \leq \psi(x)\psi(f(x)), \forall x \in X.$$

*Then f has a fixed point in X. Moreover, for any $x_0 \in X$ the sequence $f^n(x_0)$ converges to a fixed point of f.*

*Proof.* Choose $x_0 \in X$. Due the above condition, the sequence $\psi(f^n(x_0))$ is decreasing, and thus convergent. Reasoning as in the proof of the Caristi theorem, we get that $f^n(x_0)$ admits a limit $x \in X$, for X is complete. The continuity of f then entails.

$$f(x) = \lim_n f(f^n(x_0)) = x.$$

□

We conclude with the following extension of Theorem 4.6, that we state without proof.

**Theorem 5.4** (*Ciric*). *Let $X$ be a complete metric space, and let $f : X \to X$ be such that*

$$d(f(x_1), f(x_2))$$

$$\leq \lambda max d(x_1, x_2), d(x_1, f(x_1)), d(x_2, f(x_2)), d(x_1, f(x_2)), d(x_2, f(x_1))$$

*for some $\lambda < 1$ and every $x_1, x_2 \in X$. Then $f$ has a unique fixed point $x \in X$. Moreover, $d(f^n(x_0), x) = \boldsymbol{O}(\lambda^n) for any x_0 \in X$.*

Also in this case f need not be continuous. However, it is easy to check that it is continuous at the fixed point. The function g of the former example fulfills the hypotheses of the that with $\lambda = \frac{2}{3}$.

## 5.1 WEAK CONTRACTION

We now dwell on the case of maps on a metric space which are contractive without being contractions.

**Definition 5.1.** *Let $X$ be a metric space with a distance $d$. A map $f : X \to X$ is a weak contraction if*

$$d(f(x_1), f(x_2)) < d(x_1, x_2), \forall x_1 \neq x_2 \in X.$$

### 5.1.1 FIXED POINT THEOREMS

Being a weak contraction is not in general a sufficient condition for f in order to have a fixed point, as it is shown in the following simple example.

**Example 11.** *Consider the complete metric space $X = [1, +\infty)$, and let $f : X \to X$ be defined as $f(x) = (x + 1)/x$.*
*It is easy to see that $f$ is a weak contraction with no fixed points. Nonetheless, the condition turns out to be sufficient when $X$ is compact.*

**Theorem 5.5.** *Let $f$ be a weak contraction on a compact metric space $X$. Then $f$ has a unique fixed point $x \in X$. Moreover, for any $x_0 \in X$ the sequence $f^n(x_0)$ converges to $x$ .*

*Proof.* The uniqueness argument goes exactly as in the proof of Theorem 1.3. From the compactness of $X$, the continuous function $x \mapsto d(x, f(x))$ attains its minimum at some $x \in X$. If $x \neq f(x)$, we get

$$d(x, f(x)) = min_{y \in X} d(y, f(y)) \leq d(f(x), f(f(x))) < d(x, f(x))$$

which is impossible. Thus $x$ is the unique fixed point of $f$ (and so of $f^n$ for all $n \geq 2$).

Let now $x_0 \neq x$ be given, and define $d_n = d(f^n(x_0), x)$. Observe that

$$d_n + 1 = d(f^n + 1(x_0), f(x)) < d(f^n(x_0), x) = d_n.$$

Hence $d_n$ is strictly decreasing, and admits a $\lim r \geq 0$.

Let now $f^{n_k}(x_0)$ be a subsequence of $f^n(x_0)$ converging to some $z \in X$. Then

$$r = d(z, x) = \lim_{k \to \infty} d_{n_k} = \lim k \to \infty d_{n_k + 1} = \lim_{k \to \infty} d(f(f^{n_k}(x0)), x) = d(f(z), x).$$

But if $z \neq x$, then $d(f(z), x) = d(f(z), f(x)) < d(z, x)$. Therefore any convergent subsequence of $f^n(x_0)$ has limit $x$, which, along with the compactness of X, implies that converges to $x$ Obviously, we can relax the compactness of X by requiring that f(X) be compact (just applying the theorem on the restriction of f on f(X)). □

Arguing like in Corollary 1.5, it is also immediate to prove the following

**Corollary 5.5.1.** *Let $X$ be a compact metric space and let $f : X \to X$. If $f^n$ is a weak contraction, for some $n \geq 1$, then $f$ has a unique fixed point $x \in X$.*

### 5.1.2 A converse to the contraction principle

Assume we are given a set $X$ and a map $f : X \to X$. We are interested to find a metric d on X such that $(X, d)$ is a complete metric space and $f$ is a contraction on X. Clearly, in light of Theorem 1.3, a necessary condition is that each iterate $f^n$ has a unique fixed point. Surprisingly enough, the condition turns out to be sufficient as well.

**Theorem 5.6** (Bessaga)**.** *Let $X$ be an arbitrary set, and let $f : X \to X$ be a map such that $f^n$ has a unique fixed point $x \in X$ for every $n \geq 1$. Then for every $\epsilon \in (0, 1)$, there is a metric $d = d_\epsilon$ on $X$ that makes $X$ a complete metric space, and $f$ is a contraction on $X$ with Lipschitz constant equal to $\epsilon$.*

*Proof.* Choose $\epsilon \in (0, 1)$. Let $Z$ be the subset of $X$ consisting of all elements $z$ such that $f^n(z) = x$ for some $n \in N$.

We define the following equivalence relation on $X\ Z$ : we say that $xy$ if and only if $f^n(x) = f^m(y)$ for some $n, m \in N$.

Notice that if $f^n(x) = f^m(y)$ and $f^{n'}(x) = fm'(y)$ then $f^{n+m'}(x) = f^{m+n'}(x)$. But since $x \notin Z$, this yields $n + m' = m + n'$ , that is, $nm = n' - m'$ . At this point, by means of the axiom of choice, we select an element from each equivalence class. We now proceed defining the distance of $x$ from a generic $y \in X$ by setting $d(x, x) = 0, d(y, x) = \epsilon n$ if $y \in Z$ with $y \neq x$, where

$$n = minm \in N : fm(y) = x, andd(y, x) = \epsilon^{nm}$$

if $y \notin Z$, where $n, m \in N$ are such that $f^n(x') = f^m(x)$, $x'$ being the selected representative of the equivalence class $[x]$. The definition is unambiguous, due to the above discussion. Finally, for any $z, y \in X$, we set

$$d(z, y) = \begin{cases} d(z, x) + d(y, x) & ifz \neq y \\ 0 & ifx = y \end{cases}$$

It is straightforward to verify that $d$ is a metric. To see that $d$ is complete, observe that the only Cauchy sequences which do not converge to $x$ are ultimately constant. We are left to show that $f$ is a contraction with Lipschitz constant equal to $\epsilon$. Let $y \in X, y \neq x$. If $y \in Z$ we have

$$d(f(y), f(x)) = d(f(y), x) \leq \epsilon^n = \epsilon\epsilon^{(n+1)} = \epsilon d(y, x).$$

If $y \notin Z$ we have

$$d(f(y), f(x)) = d(f(y), x) = \epsilon^{nm} = \epsilon\epsilon^{n(m+1)} = \epsilon d(y, x)$$

since $yf(y)$. The thesis follows directly from the definition of the distance $\square$

#### 5.1.3 Fixed points of non-expansive maps

Let $X$ be a Banach space, $C \subset X$ nonvoid, closed, bounded and convex, and let $f : C \to C$ be a non-expansive map. The problem is whether f admits a fixed point in C. The answer, in general, is false.

**Example 12.** *Let $X = c_0$ with the supremum norm. Setting $C = \bar{B}_X(0, 1)$, the map $f : C \to C$ defined by*

$$f(x) = (1, x_0, x_1, \cdots), \qquad forx = (x_0, x_1, x_2, \cdots) \in C$$

*is non-expansive but clearly admits no fixed points in C.*

**Theorem 5.7** (Browder-Kirk)**.** *Let $X$ be a uniformly convex Banach space and $C \subset X$ be nonvoid, closed, bounded and convex. If $f : C \to C$ is a non-expansive map, then f has a fixed point in C.*

We provide the proof in the particular case when X is a Hilbert space (the general case may be found)

*Proof.* Let $x \in C$ be fixed, and consider a sequence $r_n \in (0, 1)$ converging to 1. For each $n \in N$, define the map $f_n : C \to C$ as

$$f_n(x) = r_n f(x) + (1r_n)x.$$

Notice that $f_n$ is a contractions on C, hence there is a unique $x_n \in C$ such that $f_n(x_n) = x_n$. Since C is weakly compact,$x_n$ has a subsequence (still denoted by $x_n$) weakly convergent to some $x \in C$. We shall prove that x is a fixed point of f. Notice first that

$$\lim_{n\to\infty}(||f(x)x_n||^2||xx_n||^2) = ||f(x)x||^2$$

Since f is non-expansive we have

$$\begin{aligned} ||f(x)x_n|| &\leq ||f(x)f(x_n)|| + ||f(x_n)x_n|| \\ &\leq ||xx_n|| + |f(x_n)x_n|| \\ &= ||xx_n|| + (1r_n)||f(x_n)x||. \end{aligned}$$

But $r_n \to 1$ as $n \to \infty$ and C is bounded, so we conclude that $\lim_{n\to\infty} sup(||f(x)x_n||^2||x$
0

which yields the equality $f(x) = x$ □

**Proposition 5.1.** *In the hypotheses of above Theorem, the set F of fixed points of f is closed and convex.*

*Proof.* The first assertion is trivial. Assume then $x_0, x_1 \in F$, with $x_0 \neq x_1$, and denote $x_t = (1t)x_0 + tx_1$, with $t \in (0, 1)$. We have

$$||f(x_t)x_0|| = ||f(x_t)f(x_0)|| \neq ||x_tx_0|| = t||x_1x_0||$$

$$||f(x_t)x_1|| = ||f(x_t)f(x_1)|| \leq ||x_tx_1|| = (1t)||x_1x_0||$$

that imply the equalities

$$||f(x_t)x_0|| = t||x_1x_0||$$

$$||f(x_t)x_1|| = (1t)||x_1x_0||.$$

The proof is completed if we show that $f(x_t) = (1t)x_0 + tx_1$. This follows from a general fact about uniform convexity, which is recalled in the next lemma □

**Lemma 5.8.** *Let X be a uniformly convex Banach space, and let $\alpha, x, y \in X$ be such that*

$$||\alpha x|| = t||xy||, \qquad ||\alpha y|| = (1t)||xy||,$$

*for some $t \in [0,1]$. Then $\alpha = (1t)x + ty$.*

*Proof.* Without loss of generality, we can assume $t \geq 1/2$. We have

$$\begin{aligned} ||(1t)(\alpha x)t(\alpha y)|| &= ||(12t)(\alpha x)t(xy)|| \\ &\geq t|||xyk(12t)||\alpha x|| \\ &= 2t(1t)||xy||. \end{aligned}$$

Since the reverse inequality holds as well, and

$$(1t)||\alpha x|| = t||\alpha y|| = t(1t)||xy||$$

from the uniform convexity of X (but strict convexity would suffice) we get

$$||\alpha(1t)xty|| = ||(1t)(\alpha x) + t(\alpha y)|| = 0$$

as claimed □

#### 5.1.4 The Riesz mean ergodic theorem

If T is a non-expansive linear map of a uniformly convex Banach space, then all the fixed points of T are recovered by means of a limit procedure.

**Definition 5.2.** *Projections- Let X be a linear space. A linear operator $P : X \to X$ is called a projection in X if $P^2x = PPx = Px$ for every $x \in X$.*

**NOTE** P is the identity operator on ran(P), and the relations ker(P) = ran($I$  P), ran(P) = ker(I  P) and $ker(P) \cap ran(P) = 0$ hold. Moreover every element $x \in X$ admits a unique decomposition $x = y + zwithy \in ker(P)andz \in ran(P)$.

**Proposition 5.2.** *If X is a Banach space, then a projection P is continuous if and only if $X = ker(P) \oplus ran(P)$.*

(The notation $X = A \oplus B$ is used to mean that A and B are closed subspaces of X such that $A \cap B = 0 and A + B = X$. )

*Proof.* If P is continuous, so is I P. Hence ker(P) and ran(P) = ker(I P) are closed.

Conversely, let $x_n \to x$, and $Px_n \to y$. Since ran(P) is closed, $y \in ran(P)$, and therefore $Py = y$. But $Px_n x_n \in ker(P)$, and ker(P) is closed. So we have $xy \in ker(P)$, which impliesv$Py = Px$. From the closed graph theorem, P is continuous □

**Theorem 5.9** (F. Riesz)**.** *Let X be a uniformly convex Banach space. Let* $T : X \to X$*be a linear operator such that*

$$||Tx|| \leq ||x||, \forall x \in X$$

*Then for every* $x \in X$ *the limit*

$$p_x = \lim_{n \to \infty} \frac{x + T^x + \cdots + T^n x}{n+1}$$

*exists. Moreover, the operator* $P : X \to X$ *defined by* $Px = p_x$ *is a continuous projection onto the linear space* $M = y \in X : Ty = y$.

*Proof.* Fix $x \in X$, and set

$$C = co(x, Tx, T^{\bar{2}}x, T^3x, \cdots)$$

C is a closed nonvoid convex set, and from the uniform convexity of X there is a unique $p_x \in C$ such that

$$\mu = ||px|| = inf||z|| : z \in C$$

Select $\epsilon > 0$. Then, for $p_x \in C$, there are $m \in N$ and nonnegative constants $\alpha_0, \alpha_1, \cdots,\ alpha_m$ with $\sum_{j=0}^{m} \alpha_j = 1$such that, setting

$$z = \sum_{j=0}^{m} \alpha_j T^j x$$

$$thereholds$$

——$p_x z|| < \epsilon$

In particular, for every $n \in N$,

$$\left|\left|\frac{z + Tz + \cdots + T^n z}{n+1}\right|\right| \leq ||z|| \leq \mu + \epsilon.$$

Notice that

$z + Tz + \cdots + T^n z = (\alpha_0 x + \cdots + \alpha_m T^m x) + (\alpha_0 Tx + \cdots + \alpha_m T^{m+1} x) + \cdots + (\alpha_0 T^n x + \cdots + \alpha_m$

Thus, assuming n is strictly greater than m,we get

$$z + Tz + \cdots + T^n z = x + Tx + \cdots + T^n x + r$$

where

$r = (\alpha_0 1)x + \cdots + (\alpha_0 + \alpha_1 + \cdots + \alpha_{m1} 1)T^{m1} x + (1\alpha_0)T^{1+n} x + \cdots + (1\alpha_0 \alpha_1 \cdots \alpha_{m1})T^{m+n} x.$

Therefore

$$\frac{x + Tx + \cdots + T^n x}{n+1} = \frac{z + Tz + \cdots + T^n z}{n+1} - \frac{r}{n+1}$$

Since

$$\left|\left|\frac{r}{n+1}\right|\right| \leq \frac{2m||x||}{n+1}$$

upon choosing n enough large such that $2m||x|| < \epsilon(n+1)$ we have

$$\left|\left|\frac{x + Tx + \cdots + T^n x}{n+1}\right|\right| \leq \left|\left|\frac{z + Tz + \cdots + T^n z}{n+1}\right|\right| + \left|\left|\frac{r}{n+1}\right|\right| \leq \mu + 2\epsilon$$

On the other hand, it must be

$$\left|\left|\frac{x + Tx + \cdots + T^n x}{n+1}\right|\right| \geq \mu$$

Then we conclude that

$$\lim_{n\to\infty} \left|\left|\frac{x + Tx + \cdots + T^n x}{n+1}\right|\right| = \mu$$

This says that the above is a minimizing sequence in C, and due to the uniform convexity of X, we gain the convergence

$$\lim_{n\to\infty} \frac{x + Tx + \cdots + T^n x}{n+1} = p_x$$

We are left to show that the operator $Px = p_x$ is a continuous projection onto $M$. Indeed, it is apparent that if $x \in M$ then $p_x = x$.

In general,

$$Tp_x = \lim_{n\to\infty} \frac{Tx + T^2 x + \cdots + T^{+1} nx}{n+1} = p_x + \lim_{n\to\infty} \frac{T^{n+1} xx}{n+1} = p_x.$$

Finally, $P^2 x = PPx = Pp_x = p_x = Px$. The continuity is ensured by the relation $||p_x|| \leq ||x||$.

□

When X is a Hilbert space, P is actually an orthogonal projection. This follows from the next proposition.

**Proposition 5.3.** *Let H be a Hilbert space,* $P = P^2 : H \to H$ *a bounded linear operator with* $||P|| \leq 1$*. Then P is an orthogonal projection.*

*Proof.* Since P is continuous, ran(P) is closed. Let then E be the orthogonal projection having range ran(P). Then

$$P = E + P(IE)$$

. Let now $x \in ran(P)^{\perp}$.

For any $\epsilon > 0$ we have

$$||P(Px + \epsilon x)|| \leq ||Px + \epsilon x||$$

which implies that

$$||Px||^2 \leq \frac{\epsilon}{2+\epsilon} ||x||^2$$

Hence $Px = 0$, and the equality P = E holds □

The role played by uniform convexity in last Theorem is essential, as the following example shows.

**Example 13.** *Let $X = l^\infty$, and let $T \in L(X)$ defined by*

$$Tx = (0, x_0, x_1, \cdots), forx = (x_0, x_1, x_2, \cdots) \in X$$

*. Then $T$ has a unique fixed point, namely, the zero element of $X$. Nonetheless, if $y = (1, 1, 1, \cdots)$, for every $n \in N$ we have*

$$\left|\left|\frac{y + Ty + \cdots + T^n y}{n+1}\right|\right| = \frac{||(1, 2, \cdots, n, n+1, n+1, \cdots)||}{n+1} = 1.$$

## 5.2 The Brouwer fixed point theorem

**Definition 5.3.** *Let $D^n = x \in \Re^n : ||x|| \leq 1$. A subset $E$ of $D^n$ is called a retract of $D^n$ if there exists a continuous map $r : D^n \to E$(called retraction) such that $r(x) = x$ for every $x \in E$*

**Lemma 5.10.** *The set $S^{n1} = x \in R^n : ||x|| = 1$is not a retract of $D^n$ .*

**Theorem 5.11** (Brouwer)**.** *Let $f : D^n \to D^n$ be a continuous function. Then $f$ has a fixed point $x \in D^n$*

## 5.3

**Theorem 5.12** (Fundamental of algebra)**.** *Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree $n \geq 1$. Then there exits $z_0 \in C$ such that $p(z_0) = 0$.*

*Proof.* For our purposes, let us identify C with $R^2$.

Suppose without loss of generality $a_n = 1$. Let $r = 2 + |a_0| + \cdots + |a_{n1}|$.

Define now the continuous function $g : C \to C$ as

$$g(z) = \begin{cases} z - \frac{p(z)e^{i(1-n)\theta}}{r} & |z| \leq 1 \\ z - \frac{p(z)e^{(1-n)}}{r} & |z| > 1 \end{cases}$$

where $z \in C$ with $\theta \in [0, 2\pi)$. Consider now the compact and convex set $C = z : |z| \leq r$. In order to apply the Brouwer fixed point theorem we need to show that $g(C) \subset C$.

Indeed, if $|z| \leq 1$,

$$|g(z)| \leq |z| + \frac{|p(z)|}{r} \leq 1 + \frac{1 + |a_0| + \cdots + |a_{n1}|}{r} \leq 2 \leq r.$$

Conversely, if $1 < |z| \leq r$ we have $|g(z)| \leq \left| z - \frac{p(z)}{rz^{n-1}} \right| = \left| z - \frac{z}{r} - \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{rz^{n-1}} \right|$
$\leq r - 1 + \frac{|a_0| + \cdots + |a_{n1}|}{r} \leq r - 1 + \frac{r-2}{r} \leq r$

C is invariant for g, and so $g$ has a fixed point $z_0 \in C$, which is clearly a root of p □

# Chapter 6

# WEAK CONTRACTION PRINCIPLE IN b-METRIC SPACE

## 6.1 b-Metric Space

**Definition 6.1.** *Let $X$ be a non-empty set and $s \geq 1$ be any real number. A function $d : X \times X \to \mathcal{R}_+$ is called a b-metric provided that, for all $x,y,z \in X$*

1. $d(x,y)=0 \Leftrightarrow x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq s\Big(d(x,y) + d(y,z)\Big)$

*A pair $(X,d)$ is called b-metric space*

- *If $s=1$ , then it reduces to the usual metric space*

**Example 14.** *Let $X = l_p (0<p<1)$*

*Let $d : l_p \times l_p \to \mathcal{R}_+$ by*

$$d(x_n, y_n) = \Big( \sum_{n=1}^{\infty} \mid (x_n - y_n) \mid^p \Big)^{\frac{1}{p}} \text{ where } x_n, y_n \in l_p$$

*Let $x = x_n, y = y_n \in l_p$*

$$d(x,y) = 0 \Leftrightarrow \Big( \sum_{n=1}^{\infty} \mid (x - y) \mid^p \Big)^{\frac{1}{p}} = 0$$

$$\Leftrightarrow \Big( \sum_{n=1}^{\infty} \mid (x - y) \mid^p \Big) = 0$$

$$\Leftrightarrow \mid (x - y) \mid^p = 0$$

$\Leftrightarrow x - y = 0$

$\Leftrightarrow x = y$

*In the similar way we will get,* $d(x, y) = d(y, x)$

$$d(x, y) \leq \frac{d\Big(d(x, z) + d(z, y)\Big)}{2^p}$$

*Then* $l_p$ *is a b-metric space*

**Definition 6.2.** *Let* $(X, d)$ *be a b-metric space.Then a sequence* $(x_n) \in X$ *is called a cauchy sequence iff for all* $\epsilon > 0$*, there exist* $n(\epsilon) \in \mathcal{N}$ *such that for each* $m \geq n(\epsilon)$*,* $d(x_n, y_n) < \epsilon$

**Definition 6.3.** *Let* $(X, d)$ *be a b-metric space.Then a sequence* $(x_n)$ *in* $X$ *is convergent iff for all* $\epsilon > 0$*, there exist* $x \in X$ *such that* $d(x_n, x) < \epsilon$ *whenever* $n \geq n(\epsilon)$ *where* $n(\epsilon) \in \mathcal{N}$

**Definition 6.4.** *The b-metric space is complete if every cauchy sequence in it is convergent*

## 6.2 Result

**Theorem 6.1.** *Let* $(X, d, s)$ *be a complete b-metric space and* $s \geq 1$ *be a given real number.Let* $F : X \to X$ *be a mapping such that*

$$d\Big(F(x), F(y)\Big) \leq d(x, y) - \Phi\Big(d(x, y)\Big)$$

*where* $\Phi : \mathcal{R}_+ \to \mathcal{R}$ *is a function such that* $lim_{n\to\infty}\Big(inf\Phi(t_n)\Big) > (s-1)l$ *whenever* $limsup(t_n) \geq l > 0$ *, then* $F$ *has a unique fixed point*

*Proof.* Let $x_0 \in X$ be any element.

We construct a sequence $(x_n)$ by $x_n = F(x_{n-1})$ , for $n \geq 1$

Then

$$d(x_n, x_{n+1}) = d(F(x_{n-1}), F(x_n))$$

$$\leq d(x_{n-1}, x_n) - \Phi\Big(d(x_{n-1}, x_n)\Big)$$

$$\Rightarrow d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$$

It follows that $d(x_{n-1}, x_n)$ is a monotone decreasing sequence of non-negative real number and hence

$$d(x_{n-1}, x_n) \to lasn \to \infty \tag{6.1}$$

Since $l > 0$,

$$lim_{n\to\infty}inf\Big(\Phi(d(x_{n-1}, x_n))\Big) > 0$$

Taking limit on (1),

$$l \leq l - lim_{n\to\infty}inf\Big(\Phi(d(x_{n-1}, x_n))\Big)$$

which is a contradiction

$$hencelim_{n\to\infty}d(x_{n-1}, x_n) = 0$$

Now we prove that $(x_n)$ is a cauchy sequence .

Suppose $(x_n)$ is not cauchy sequence.

Then there exist $\epsilon > 0, k > 0$ for which there exists two sequences $(x_{m_k})$ and $(x_{n_k})$ with $n_k > m_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon$$

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon$$

Then for all $k > 0$,

$$\epsilon \leq d(x_{m_k}, x_{n_k})$$

$$< s\Big(d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k})\Big)$$

$$\leq s\epsilon + sd(x_{n_{k-1}}, x_{n_k})$$

Taking limit infimum ,

$$\epsilon \leq lim_{k\to\infty} inf\Big(d(x_{m_k}, x_{n_k})\Big)$$

$$\leq lim_{k\to\infty} sup\Big(d(x_{m_k}, x_{n_k})\Big)$$

$$\leq s\epsilon$$

For all $k > 0$,

$$d(x_{m_{k-1}}, x_{n_{k-1}}) \leq s\Big(d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_k}, x_{n_{k-1}})\Big)$$

$$\leq sd(x_{m_{k-1}}, x_{m_k}) + s^2\Big(d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k-1}})\Big)$$

and,

$$d(x_{m_k}, x_{n_k}) \leq s\Big(d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{n_k})\Big)$$

$$\leq sd(x_{m_k}, x_{m_{k-1}}) + s^2\Big(d(x_{m_{k-1}}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k})\Big)$$

Taking infimum,

$$\frac{\epsilon}{s^2} \leq lim_{k\to\infty} inf(d(x_{m_{k-1}}, x_{n_{k-1}}))$$

$$\leq lim_{k\to\infty} sup(d(x_{m_{k-1}}, x_{n_{k-1}}))$$

Put $x = x_{m_k}$ and $y = x_{n_{k-1}}$,then

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_{k-1}}, x_{n_{k-1}}) - \Phi\Big(d(x_{m_{k-1}}, x_{n_{k-1}})\Big)$$

then

$$\Phi\Big(d(x_{m_{k-1}}, x_{n_{k-1}})\Big) \leq d(x_{m_{k-1}}, x_{n_{k-1}}) - d(x_{m_k}, x_{n_k})$$

Taking limit supremum,

$$lim_{k\to\infty}sup\Big(\Phi(d(x_{m_{k-1}}, x_{n_{k-1}}))\Big) \leq lim_{k\to\infty}sup(d(x_{m_{k-1}}, x_{n_{k-1}})) - lim_{k\to\infty}inf(d(x_{m_{k-1}}$$

then, $lim_{k\to\infty}sup\Big(\Phi(d(x_{m_{k-1}}, x_{n_{k-1}}))\Big) > (s-1)\epsilon$

which is a contradiction.

therefore $(x_n)$ is a cauchy sequence and hence $(x_n) \to x \in X$ ,since $(X, d)$ is complete.Then

$$\begin{aligned} d(x, F(x)) &\leq s\Big(d(x, x_{n+1}) + d(x_{n+1}, F(x))\Big) \\ &= s\Big(d(x, x_{n+1}) + d(F(x_n), F(x))\Big) \\ &= s(d(x, x_{n+1}) + s^2\Big(d(x_n, x) + \Phi(d(x_n, x))\Big) \\ &\leq s(d(x, x_{n+1})) + s^2(d(x_n, x)) \end{aligned}$$

Taking $n \to \infty$, $d(x, F(x)) = 0$ i.e, $x = F(x)$ If $x$ and $y$ are two fixed points of $F$ then $d(x, y) > 0$ and

$$\begin{aligned} d(x, y) &= d(F(x), F(y)) \\ &= d(x, y) - \Phi(d(x, y)) \\ &\leq d(x, y) \end{aligned}$$

which is a contradiction. Hence $x = y$

$\Rightarrow$ Fixed points are unique. □

**Example 15.** *Let x=[0,1] be equipped with the b-metric* $d(x, y) =\mid x - y \mid^2$ *for all x,y* $\in X$ *,clearly d is a metric on d.*

*Then* $(X,d)$ *is a b-metric space with parameters* $s = 2$*.Also* $(X,d)$ *is complete metric space,since [0,1] is closed and bonded set.*

*Let* $F : X \to X$ *as,*

$$F(x) = x - \frac{x^2}{2}, x \in [0,1]$$

*and* $\Phi : \mathcal{R}_+ \to \mathcal{R}$ *by,*

$$\Phi(t) = \frac{t^2}{2} t \in [0,1]$$

*Then for* $x, y \in X$*,*

$$\begin{aligned} d(F(x), F(y)) &= \mid F(x) - F(y) \mid^2 \\ &= \mid (x - \frac{x^2}{2}) - (y - \frac{y^2}{2}) \mid^2 \\ &\leq \mid (x - y) - (\frac{x^2}{2} - \frac{y^2}{2}) \mid^2 \end{aligned}$$

$$\leq *d(x - y) - \Phi(d(x,y))$$

*F has a unique fixed point in [0,1]*

$$F(x) = x \Rightarrow x - \frac{x^2}{2} = x$$

$$\Rightarrow \frac{x^2}{2} = 0$$

$$\Rightarrow x = 0$$

$x = 0$ *is the unique fixed point of* $F$

# Chapter 7

# CO-CYCLIC WEAK CONTRACTION AND FIXED POINTS

## 7.1 Cyclic Representation

**Definition 7.1.** *Let $X$ be a non-empty set,$m$ a positive integer and $F : X \to X$ a self-map.$X = \bigcup_{i=1}^{m} A_i$ is said to be a cyclic representation of $X$ with respect to the map $F$ if the following hold:*

1. $A_i, i = 1, 2, ..., m$ *are nonempty subset of* $X$
2. $F(A_1) \subset A_2, F(A_2) \subset A_3, ......., F(A_{m-1}) \subset A_m, F(A_m) \subset A_1$

**Example 16.** *Let X=[0,2] , $A_1$=[0,1] ,$A_2 = [\frac{1}{2}, \frac{3}{2}]$ , $A_3 = [1, 2]$*
*Define a self map $F$ on $X$ by,*

$$F(x) = x + \frac{1}{x} if x \in [0, \frac{1}{2}]$$

$$F(x) = 1 if x \in (\frac{1}{2}, \frac{3}{2}]$$

$$F(x) = x - 1 if x \in (\frac{3}{2}, 2]$$

*Then,*

$$F(A_1) = [\frac{1}{2}, 1] \subset A_2$$

$$F(A_2) = 1 \subset A_3$$

$$F(A_3) = 1 \subset A_1$$

*therefore* $X = \bigcup_{i=1}^{3} A_i$ *ia a cyclic representation of* $X$ *with respect to* $F$

**Theorem 7.1.** *Let* $X$ *be a compact metric space ,*$m$ *a positive integer and* $F : X \to X$ *a continuous operator.suppose that* $A_1$, $A_2$,......$A_m$ *are nonempty subsets of* $X$ , $X = \bigcup_{i=1}^{m} A_i$ *satisfying:*

1. $X = \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* $X$ *with respect to* $F$
2. $d(F(x), F(y)) \leq d(x, y) - \Phi(d(x, y))$ *for any* $x \in A_i$ *and* $y \in A_{i+1}$ *where* $\Phi \in \mathcal{F}$ *where* $\mathcal{F}$=$\Big\{\Phi : \mathcal{R}_+ \to \mathcal{R}_+$ *such that* $\Phi$ *is nondecreasing ,* $\Phi(0) = 0$,$\Phi(t) > 0$ *for* $t > 0\Big\}$

*Then* $F$ *has a unique fixed point .*

**Definition 7.2.** *Let* $X$ *be a nonempty.Two self maps* $F$,$G$ ; $F, G : X \to X$ *are said be weakly compatible if then commute at their coincidence points.If* $x \in X$ *such that* $F(x) = G(x)$,*then* $(FG)(x) = (GF)(x)$

### 7.2 Co-cyclic Representation

**Definition 7.3.** *Let* $X$ *be a nonempty set,*$m$ *a positive integer and* $F, G : X \to X$ *be two self maps.*$X = \bigcup_{i=1}^{m} A_i$ *is said be a co-cyclic representation of* $X$ *between* $F$ *and* $G$ *if:*

1. $A_i$, $i$ =1,2,....$m$ *are nonempty subset of* $X$

2. $G(A_1) \subset F(A_2)$,$G(A_2) \subset F(A_3)$,.........,$G(A_m) \subset F(A_1)$

**Example 17.** *Let* $x = [0, 1]$ *and* $A_1 = [0, \frac{1}{2}]$ *and* $A_2 = [\frac{1}{2,1}]$

*Define* $F$ *and* $G$ *on* $X$ *by ,*

$G(x) = \Big\{ x + \frac{1}{2}$ *if* $x \in [0, \frac{1}{2}]$ , $1 - x$ *if* $x \in (\frac{1}{2}, 1]$

*and* $F(x) = \Big\{ x$ *if* $x \in [0, \frac{1}{2}], 2x - 1$ *if* $x \in (\frac{1}{2}, 1]$

*Now* $G(A_1)$= *[*$\frac{1}{2}$*,1] and* $F(A_2)$=*[*$\frac{1}{2}$*,1]*

$G(A_2)$=*[0,*$\frac{1}{2}$*] and* $F(A_2)$=*[0,*$\frac{1}{2}$*]*

*Therefore* $G(A_1) = F(A_2)$ *and* $G(A_2) = F(A_1)$

$RightarrowX = \bigcup_{i=1}^{2} A_i$ *is a co-cyclic representation of* $X$ *between* $F$ *and* $G$

**Definition 7.4.** *Let* $(X, d)$ *be a metric space,*$m$ *is a positive integer,*$A_1$ *,* $A_2$ *,....*$A_m$ *a closed nonempty subsets of* $X$*, and* $X = \bigcup_{i=1}^{m} A_i$*.An operator* $G : X \to X$ *is said to be co-cyclic weak contraction if there exist an operator* $F : X \to X$ *such that*

1. $X = \bigcup_{i=1}^{m}$ *is a cyclic representation of* $X$ *between* $F$ *and* $G$
2. $d(G(x), G(y)) \leq d(F(x), F(y)) - \Phi\Big(d(F(x), F(y))\Big)$*, for any* $x \in A_i$ *and* $y \in A_{i+1}$ *where* $A_{m+1} = A_1$ *and* $\Phi \in \mathcal{F}$

**Theorem 7.2.** *Let* $(X, d)$ *be a compact metric space and* $F, G : X \to X$ *be two continuous operators.Suppose that* $m$ *is a positive integer,*$A_1$*,* $A_2$*,.....,*$A_m$ *are nonempty subsets of* $X$ *and* $X = \bigcup_{i=1}^{m} A_i$ *satisfying,*

1. $X = \bigcup_{i=1}^{m}$ *is a cyclic representation of* $X$ *between* $F$ *and* $G$
2. $d(G(x), G(y)) \leq d(F(x), F(y)) - \Phi\Big(d(F(x), F(y))\Big)$*, for any* $x \in A_i$ *and* $y \in A_{i+1}$ *where* $A_{m+1} = A_1$ *and* $\Phi \in \mathcal{F}$

*If the pair of operators* $(F, G)$ *are weakly compatible on* $X$ *then* $F$ *and* $G$ *have a unique common fixed point in* $X$*.*

*Proof.* Let $x_0$

$\in X$

Since $G(A_i) \subset F(A_{i+1})$ for each $i$=1,2,3,...,$m$ and $G(A_m) \subset F(A_1)$, there exists $x_1 \in A$ such that $G(x_0) = F(x_1)$

On continuing the process, inductively we get a sequence $x_n \in X$ such that

$$G(x_n) = F(x_{n+1}) \text{ for each } n\text{=0,1,2....}$$

If there exist $n_0 \in \mathcal{N}$ with $G(x_{n_0+1}) = G(x_{n_0}) = F(x_{n_0+1})$ and thus $F$ and $G$ have coincidence point $x_{n_0+1}$

Suppose that $x_{n+1} \neq x_n \forall n$=0,1,2....

We have to show that the sequence $\{d(F(x_n), F(x_{n+1}))\}$ is a non increasing sequence.

therefore for each $n > 0$ ,there exist $i_n \in \{1, 2, 3, ...m\}$ such that $x_{n-1} \in A_{i_n-1}$ and $x_n \in A_{i_n}$ and,

$$\begin{aligned} d(F(x_n), F(x_{n+1})) &= d(G(x_{n-1}), G(x_n)) \\ &\leq d(F(x_{n-1}), F(x_n)) - \Phi\Big(d(F(x_{n-1}), F(x_n))\Big) \\ &\leq d(F(x_{n-1}), F(x_n)) \text{ for each } n\text{=0,1,2....} \end{aligned}$$

Hence $\{d(F(x_n), F(x_{n+1}))\}$ is a non-increasing sequence of non negative real and hence converges to a limit $l > 0$.

Letting $n \to \infty$ in the above inequality

$$l \leq l - lim_{n\to\infty}\Phi\Big(d(F(x_n), F(x_{n+1}))\Big) \leq l$$

and hence,

$$lim_{n\to\infty}\Phi\Big(d(F(x_n), F(x_{n+1}))\Big) = 0$$

Claim: $l = 0$

Suppose $l > 0$

Since $l = inf\{d(F(x_n), F(x_{n+1})) : n \in \mathcal{N}\}$

$0 < l < d(F(x_n), F(x_{n+1}))$ for $n$ =0,1,2,.... and since $\Phi$ is non decreasing and $\Phi(t) > 0$ for $t \in (0, \infty)$

We get,

$$0 < \Phi(l) \leq \Phi\Big(d(F(x_n), F(x_{n+1}))\Big) \text{ for } n = 0,1,2,....$$

and hence letting $n \to \infty$,

$$0 < \Phi(l) \leq lim_{n\to}\Phi\Big(d(F(x_n), F(x_{n+1}))\Big)$$

which is a contradiction , therefore $l = 0$

Hence ,

$$lim_{n\to\infty}(d(F(x_n), F(x_{n+1})) = 0$$

Since $G(x_n) = F(x_{n+1})$ for each $n$ =1,2,3,.....

$$inf\{(d(F(x), G(x)) : x \in X\} = 0$$

Since the mapping $X$ to $\mathcal{R}^+$ defined by $x \mapsto d(F(x), G(x))$ is continuous and $X$ is compact.

Therefore We can find $u \in X$ such that

$$(d(F(u), G(u)) = inf\{(d(F(u), G(u)) : x \in X\}$$

But $(d(F(u), G(u)) = 0$

Hence $F(u) = G(u) = z$ (say)

i.e, $(F, G)$ has a point of coincidence.

Since $(F, G)$ is weakly compatible,

$$\begin{aligned} G(z) &= G(F(u)) \\ &= F(G(u)) \\ &= F(z) \end{aligned}$$

Hence $G(z) = F(z)$

Claim : $z = G(z)$

Suppose $z \neq G(z)$

Then $d(z, G(z)) = d(G(u), G(z))$

$$\leq d(F(u), F(z)) - \Phi\Big(d(F(u), F(z))\Big)$$

$$\leq d(z, G(z)) - \Phi\Big(d(z, G(z))\Big)$$

$$\Rightarrow \Phi\Big(d(z, G(z))\Big) \leq 0$$

But $\Phi\Big(d(z, G(z))\Big) \geq 0$

Hence $\Phi\Big(d(z, G(z))\Big) = 0$

Since $\Phi \in \mathcal{F}$,

$$d(z, G(z)) = 0$$

$$\Rightarrow G(z) = z$$

therefore $G(z) = z = F(z)$

Now we have to prove the uniqueness of fixed point.for that,

Suppose $z$ and $w$ are common fixed points of $F$ and $G$.

Since $X = \bigcup_{i=1}^{m} A_i$ is co-cyclic representation of $X$ between $F$ and $G$,

We have $z, w \in \bigcap_{i=1}^{m} A_i$

Therefore
$$\begin{aligned} d(z, w) &= d(G(z), G(w)) \\ &\leq d(F(z), F(w)) - \Phi\Big(d(F(z), F(w))\Big) \\ &\leq d(z, w) - \Phi(d(z, w)) \end{aligned}$$

therefore $\Phi(d(z, w)) = 0$

Sice $\Phi \in \mathcal{F}$,

$$d(z, w) = 0$$

$$\Rightarrow z = w$$

Hence the result. □

- Note that if we choose $F = I_x$, the identity map(which is weakly compatible) in the above theorem,then the theorem gives us the unique fixed point of $G$

**Corollary 7.2.1.** *Let $(X, d)$ be a compact metric space and $G : X \to X$ be a continuous operator.Suppose that $m$ is a positive integer,$A_1$ , $A_2$ ,....., $A_m$ are nonempty subsets of $X$ and $X = \bigcup_{i=1}^{m} A_i$ satisfying:*

1. *$X = \bigcup_{i=1}^{m}$ is a cyclic representation of $X$ with respect to $G$*

2. $d(G(x), G(y)) \leq d(x, y) - \Phi(d(x, y))$, *for any* $x \in A_i$ *and* $y \in A_{i+1}$ *where* $A_{m+1} = A_1$ *and* $\Phi \in \mathcal{F}$

*Then* $G$ *has a unique fixed point in* $X$.

# Conclusion

In this project we have discussed about the theory of fixed point and theorems related to the existence and properties of fixed point. We have also discussed about the two major areas of fixed point theory , one is the contraction type mappings on complete metric spaces and second one is the fixed point theory on continous operators on compact and convex subsets of a normed space. Banach Contraction Principle , the fundamental principle in the field of functional analysis is also discussed here. We have also gone through the establishment of the extension of  Banach Contraction Principle for weak contraction mapping and even the co-cyclic representation.

# References

[1] Andrzej Granas, James Dugundji. *Fixed Point Theory*, Springer

[2] Edward Fadell, Gilles Fournier (Eds.).*Fixed Point Theory*. Quebec , Canada 1980, Springer

[3] Saleh Almezel, Qamrul Hasan Ansari, Mohamed Amine Khamsi Editors. *Topics in Fixed Point Theory,* Springer

[4] Ravi.P. Agarwal, Maria Meehan, Donal O'Regan. *Fixed Point Theory And Applications*. Cambridge University Press