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B.Sc. DEGREE (C.B.C.S.S.) EXAMINATION, MARCH 2016

Sixth Semester

Core Course—COMPLEX ANALYSIS

(For B.Sc. Mathematics Model I and II)

[2013 Admissions]

Time : Three Hours Maximum : 80 Marks

Part A (Objective Type Questions)

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Find singular points if any for the function :

   $$f(z)=\frac{z}{z^2+1}.$$

2. Define Harmonic function.
3. Define cosine function of a complex variable z.
4. What is the value of $\int_{|z|=1}\left(z^2+4\right)dz$?
5. Define a simply connected domain.
6. State Morera's theorem.
7. Write the Maclaurian series of $\frac{1}{1+z}$ if $|z|<1$.
8. Find the Laurent series of $f(z)=\frac{1}{z-1}$ valid for $|z|>1$.
9. Find the residue at $z=0$ of the function $f(z)=\frac{1}{z+z^2}$.
10. Find the residue of $f(z)=\frac{e^z}{z^2}$ at its pole.

(10 × 1 = 10)

Turn over

Part B (Short Answer Questions)

Answer any **eight** *questions.*
Each question carries 2 marks.

11. Show that $f(z) = e^x e^{-iy}$ is nowhere differentiable.

12. Show that $f(z) = \sin x \cosh y + i \cos x \sinh y$ is an entire function.

13. Show that $\text{Log}\left(i^3\right) \neq 3\text{Log}\, i$.

14. Define the hyperbolic sine and the hyperbolic cosine of a complex variable z and show that $\frac{d}{dz}\cosh z = \sinh z$.

15. Evaluate $\int_C \frac{z+2}{z}\, dz$ where C is the semi-circle $z = 2e^{i\theta}, 0 \leq \theta \leq \pi$.

16. Evaluate $\int_C \frac{dz}{z^2+4}$ where C is the positive oriented circle $|z-i| = 2$.

17. Let C denote the positively oriented boundary of the square whose sides lie along the lines : $x = \pm 2$ and $y = \pm 2$. Evaluate $\int_C \frac{z}{2z+1}\, dz$.

18. Obtain the Maclaurin's series representation of the function $f(z) = e^z$.

19. State Laurent's theorem.

20. Find the nature of the singular point at $z_0 = 0$ for the function $f(z) = \frac{\sinh z}{z^4}$.

21. Using residue theorem evaluate $\int_C \frac{dz}{z^2-1}$ where C is the positive oriented circle $|z| = 2$.

22. State Jordan's lemma.

$(8 \times 2 = 16)$

Part C (Short Essay Questions)

Answer any **six** *questions.*
Each question carries 4 *marks.*

23. Prove that $f(z) = e^z$ is differentiable everywhere in the complex plane. Also find the derivative $f'(z)$.

24. If $f(z) = u(x, y) + i\, v(x,y)$ analytic in a domain D, then prove that u and v are harmonic in D.

25. If a function $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic in a domain D, then prove that $f(z)$ is constant throughout D.

26. State and prove Cauchy's inequality.

27. State and prove Liouville's theorem.

28. Derive the Taylor series representation $\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \left(|z-1| < \sqrt{2}\right)$.

29. Show that when $0 < |z-1| < 2$, $\frac{z}{(z-1)(z-3)} = -3 \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$.

30. State and prove Cauchy's residue theorem.

31. Evaluate $\int_0^{\infty} \frac{dx}{x^4+1}$.

(6 × 4 = 24)

Part D (Essay Questions)

Answer any **two** *questions.*
Each question carries 15 *marks.*

32. (a) Derive the Cauchy-Riemann equations.

(b) Show by an example that satisfaction of the Cauchy-Riemann equations at a point is not sufficient to ensure the existence the derivative of a function $f(z)$ at that point.

33. (a) State (without proof) Cauchy Integral formula. Use it to show that the derivative of an analytic function is again analytic.

(b) State and prove the maximum modulus principle.

34. (a) State and prove Taylor's theorem.

(b) Expand $f(z) = \cos z$ into a Taylor series about the point $z_0 = \frac{\pi}{2}$.

35. Use residues to evaluate :

(a) $\int_{-\infty}^{\infty} \frac{\cos 3x}{\left(x^2+1\right)^2}\, dx$.

(b) $\int_{0}^{2\pi} \frac{d\theta}{5+4\cos\theta}$.

(2 × 15 = 30)