TB145620A **Reg. No:…………………….**

Name:……………………….

B. Sc. DEGREE (C.B.C.S.S.) EXAMINATION, OCTOBER 2016

SEMESTER V - MATHEMATICS

MAT5AA - ABSTRACT ALGEBRA

Time: Three Hours **Maximum Marks: 80**

PART A

(Short Answer Questions)

I . Answer all questions . Each question carries 1 mark.

1. The smallest non abelian group has ...elements.
2. How many proper subgroups are there for the Klein 4 group?
3. What is the order of $\sigma = (4,5)(2,3,7)$in S_8?
4. Find the number of generators of a cyclic group of order 60.
5. Define an isomorphism.
6. State True or False: $\mathbb{Z}$ is a subfield of $\mathbb{Q}$.
7. How many zero divisors does the field $\mathbb{Z}_5$ have?
8. Define a division ring.
9. State True or False: $\mathbb{Q}$ is a ideal in $\mathbb{R}$.
10. Define a quotient ring.

($10 \times 1 = 10$)

PART B

(Brief Answer Questions)

II . Answer any eight questions. Each question carries 2 marks.

11. List the elements of the subgroups $\langle 3\rangle$ and $\langle 15\rangle$ in $\mathbb{Z}_{18}$.
12. Compute the products $\tau\sigma$ and $\tau^2\sigma$ if $\sigma = \begin{pmatrix} 1 & 2 & 34 & 5 & 6 \\ 3 & 1 & 45 & 6 & 2 \end{pmatrix}$and $\tau = \begin{pmatrix} 1 & 2 & 34 & 5 & 6 \\ 2 & 4 & 13 & 6 & 5 \end{pmatrix}$.
13. Define binary operation on a set. Determine whether $*$ defined by $a * b = c$, where c is the largest integer less than the product of a and b does give a binary operation on $\mathbb{Z}^+$.
14. Prove that a group has exactly one idempotent element.
15. Show that $\mathbb{Z}_p$has no proper nontrivial subgroups if p is a prime number.
16. Prove that an isomorphism maps inverses onto inverses.
17. Show that every subgroup of an abelian group is a normal subgroup.
18. Prove that a group homomorphism $\emptyset: G \to G'$ is a one-to-one map if and only if $ker(\emptyset) = \{e\}$.
19. Find all solutions of the equation $x^3 - 2x^2 - 3x = 0$in $\mathbb{Z}_{12}$.
20. Define unit element in a ring. Find all units in the ring $\mathbb{Z} \times \mathbb{Z}$.
21. Prove that if R is a ring with unity and N is an ideal of R containing a unit then $N = R$.
22. An element of a ring R is idempotent if $a^2 = a$. Show that a division ring contains exactly two idempotent elements.

($8 \times 2 = 16$)

PART C

(Short Essay Type Questions)

III . Answer any six questions. Each question carries 4 marks.

23. Which of the permutations in S_3 are even permutations? Give the table for the alternating group A_3.
24. Let S be the set of all real numbers except -1. Define $*$ on S by $a * b = a + b + ab$. Show that $\langle S, * \rangle$ is a group.
25. Prove that a subgroup of a cyclic group is cyclic.
26. Prove that an infinite cyclic group G is isomorphic to the group $\mathbb{Z}$ of integers under addition.
27. State and prove Lagrange's theorem.
28. Define an integral domain. Prove that every field F is an integral domain.
29. If R is a ring with additive identity 0, then prove that for any $a, b \in R$
    1. $0a = a0 = 0$
    2. $a(-b) = (-a)b = -(ab)$
    3. $(-a)(-b) = ab$.
30. Show that an intersection of ideals of a ring R is again an ideal of R.
31. Show that a factor ring of a field is either the trivial ring of one element or is isomorphic to the field.

$(6 \times 4 = 24)$

PART D

(Essay Type Questions)

IV . Answer any two questions. Each question carries 15 marks.

32. Prove that no permutation of a finite set can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.
33. State and prove Cayley's theorem.
34. State and prove Fundamental Homomorphism Theorem.
35. (a) Let R be a commutative ring and let $a \in R$. Show that $I_a = \{x \in R \mid ax = 0\}$ is an ideal of R.

    (b) An element a of a ring R is nilpotent if $a^n = 0$ for some $n \in \mathbb{Z}^+$. Show that the collection of all nilpotent elements in a commutative ring R is an ideal.

$(2 \times 15 = 30)$